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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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to a curve Σ (which is only Γ displaced in a particular manner) and including infinity, and for which such points on Σ as correspond to a continuity of $\lambda'(\psi)$ are singular points, C being a simple contour enclosing Σ.

The next question that arises is whether we can obtain an analogous result for an analytic function $\phi(z)$ regular only in the angular region $|\psi| \leqslant \alpha$, where $0<\alpha<\pi$. I have been able to answer this question in the affirmative only in one case, namely, when $\alpha = \pi/2$. The other two cases, *viz.*, when $\alpha >$ or $< \pi/2$ remain still open.

I propose to consider the case $\alpha = \pi/2$ in a subsequent paper.

Lastly, in the light of foregoing theorems, I make a few observations on a theorem of Hurwitz.

2. To prove theorem I, I shall first prove the following theorem.

THEOREM II. *Let Σ be a closed curve as in Theorem I. Then the necessary and sufficient condition that an integral function $\phi(z)$ should satisfy the inequality*

$$|\phi(\rho e^{i\psi})| < e^{\rho(\lambda(\psi)+\varepsilon)}, \quad (|\psi| \leqslant \pi), \quad \ldots \quad (2.1)$$

for every positive ε, and $\rho \geqslant \rho_0(\varepsilon)$, is that $\phi(z)$ should be of the form

$$\phi(z) = \frac{1}{2\pi i}\int_C e^{uz} J(u)\, du, \quad \ldots \quad (2.2)$$

where $J(u)$ is an analytic function of u in the region lying exterior to Σ, and including infinity, C being a simple contour enclosing Σ.

Let us first obtain a geometrical representation of $\lambda(\psi)$.

Suppose, in the first instance, that $\lambda'(\psi)$ and $\lambda''(\psi)$ are continuous, and $\lambda(\psi) + \lambda''(\psi) > 0$. Let us put

$$x = \lambda(\psi)\sin\psi + \lambda'(\psi)\cos\psi\,; \quad \ldots \quad (2.3)$$

$$y = U(x(\psi)) = -\lambda(\psi).\cos\psi + \lambda'(\psi)\sin\psi \quad \ldots \quad (2.4)$$

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Then x, y are continuous functions of ψ, $x(\pi) = x(-\pi)$, and

$$\lambda(\psi) = x(\psi) \sin\psi - U(x(\psi)) \cos\psi\,; \quad \ldots \quad (2.5)$$

$$U'(x(\psi)) = \tan\psi\,; \quad \ldots \quad \ldots \quad (2.6)$$

$$U''(x(\psi)) = \sec^3\psi \,/\, \{\lambda(\psi) + \lambda''(\psi)\} \quad \ldots \quad (2.7)$$

It follows that there exists a closed curve Γ in the (x, y) plane whose Cartesian equation is $y = U(x)$. $U'(x)$ is monotonic for $|\psi| \neq \pi/2$, and becomes infinite for $|\psi| = \pi/2$, and $U''(x) >$ or < 0 according as $|\psi| >$ or $< \pi/2$. The polar tangential equation of Γ is

$$p = \lambda(\psi), \quad (|\psi| \leqslant \pi), \quad \ldots \quad \ldots \quad (2.8)$$

where p is the length of the perpendicular dropped from the origin on the tangent to Γ, whose *positive* direction makes an angle ψ with the positive x-axis. $p >$ or $<$ 0 according as the curvature at a point P and the origin are on the same side or opposite sides of the tangent at P.*

Next, suppose that $\psi = \psi_1$ is a point of discontinuity of the first kind of $\lambda'(\psi)$. Then

$$U(x(\psi_1+0)) - U(x(\psi_1-0)) = \sin\psi_1 \,\{\lambda'(\psi_1+0) - \lambda'(\psi_1-0)\}$$
$$= \tan\psi_1 \,\{x(\psi_1+0) - x(\psi_1-0)\}. \quad (2.9)$$

For x lying between $x(\psi_1 - 0)$ and $x(\psi_1 + 0)$, we may define $U(x)$ by

$$U'(x) = \tan\psi_1, \dagger \quad \ldots \quad \ldots \quad (2.10)$$

so that $U(x)$ is a continuous function of x, and $U''(x) \equiv 0$.

Thus to a discontinuity of $\lambda'(x)$ corresponds a straight piece of the curve.

Lastly, suppose $\lambda(\psi) + \lambda''(\psi) \equiv 0$ for $\psi_2 \leqslant \psi \leqslant \psi_3$. Then

$$\lambda(\psi) = a \sin\psi - b\cos\psi, \quad \ldots \quad (2.11)$$

---

* This representation of $\lambda(\psi)$ would fail, if $\lambda(\psi)$ were negative in an interval of ψ of extent π. It is also evident now that the curve given by $p = \lambda(\psi) + \delta$, where δ is a positive number, lies entirely outside the curve Γ.

† If $\psi_1 = \pm\pi/2$, then we add the line $x = \pm\lambda\,\pm(\pi/2)$ to Γ, the like signs going together.

where a and b are constants. So that to the interval $\psi_2 \leqslant \psi \leqslant \psi_3$ corresponds an arc of a point-circle at the point (a, b). Here we have a discontinuity of the first kind of $U'(x)$.

Summing up, we say that under the conditions imposed on $\lambda(\psi)$ the polar tangential equation (2.8) represents a closed curve Γ, whose Cartesian equation is $y = U'(x)$, and which may have a finite number of (i) straight pieces on it; (ii) points of discontinuity of the first kind of $U'(x)$. In special cases such as $\lambda(\psi) = \sin\psi$, and $\lambda(\psi) = |\sin\psi|$, Γ may reduce to a point-circle, or a straight line.

Now we proceed to the proof of Theorem II.

We shall first show that the condition (2.2) is *necessary*.

Let the integral function $\phi(z)$ satisfying the inequality (2.1) be $\sum_0^\infty \frac{c_r z^r}{r!}$. Then if s is real and $> k$, the maximum value of $\lambda(\psi)$, we have the formula

$$J(s) = \sum_0^\infty \frac{c_r}{s^{r+1}} = \int_0^\infty e^{-sz}\,\phi(z)\,dz, \qquad \ldots \qquad (2.12)$$

where the series $\sum_0^\infty \frac{c_r}{s^{r+1}}$ is absolutely and uniformly convergent for $|s| > k$, and vanishes at infinity.

Let

$$J(s, \psi) = \int_0^{\infty(\psi)} \phi(z)\,e^{-sz}\,dz, \qquad \ldots \qquad (2.13)$$

where the integral is taken along a radius vector making an angle ψ ($|\psi| \leqslant \pi$) with the positive real axis. Putting $z = \rho e^{i\psi}$, and $s = Re^{i\theta}$, we have

$$\phi(z)\,e^{-sz} = O\,[\exp.\,\{-\rho\,(R\cos(\theta+\psi) - \lambda(\psi) - \varepsilon)\}\,, \quad (\varepsilon > 0),$$

so that $J(s, \psi)$ represents a regular function of s in the domain $D(\psi)$ given by

$$R\cos(\theta + \psi) \geqslant \lambda(\psi) + \delta > \lambda(\psi). \qquad \ldots \qquad (2.14)$$

Now suppose we take two values of ψ, ψ' and ψ'' sufficiently near to each other. Then the domains D (ψ') and D (ψ'') will have a common portion, and for a domain of values of s within this portion, the expressions J (s, ψ') and J (s, ψ'') will be equal.

For

$$|\rho e^{i\psi} \phi (\rho e^{i\psi}) e^{-R\rho e^{i(\theta+\psi)}}| \to 0, \text{ as } \rho \to \infty,$$

if s lies in the region R $\cos(\theta + \psi) > k_1$, ψ varying from ψ' to ψ'', k_1 being the maximum value of $\lambda(\psi)$ in the range

$$\psi' \leqslant \psi \leqslant \psi''$$

Therefore by the principle of analytic continuation, either of J (s, ψ') and J (s, ψ'') gives the analytic continuation of the other. Now introducing a set of values of ψ sufficiently near to each other between $-\pi$ and π and comparing the expressions J (s, ψ), two at a time, corresponding to these values of ψ, we arrive at the conclusion that J (s, ψ), where ψ may have *any* value between $-\pi$ and π, gives the analytic continuation of the function J (s) defined by (2.12).

It follows, therefore, that the function J (s) is regular over the domain D obtained by varying ψ from $-\pi$ to π in D (ψ) given by (2.14). Now we have to show that D is the region lying *exterior* to the envelope Σ of the lines

$$R \cos(\theta + \psi) = \lambda(\psi), \qquad (|\psi| \leqslant \pi). \tag{2.15}$$

Putting R $\cos\theta = \sigma$, R $\sin\theta = t$, and excluding, for the time being, those values of ψ for which $\lambda'(\psi)$ is discontinuous, the envelope of (2.15) is given by means of equations

$$\left.\begin{aligned} \sigma \cos\psi - t \sin\psi &= \lambda(\psi), \\ -\sigma \sin\psi - t\cos\psi &= \lambda'(\psi) \end{aligned}\right\}, \tag{2.16}$$

which at once lead to

$$-t = \sin\psi\, \lambda(\psi) + \cos\psi\, \lambda'(\psi), \tag{2.17}$$

$$-\sigma = -\cos\psi\, \lambda(\psi) + \sin\psi\, \lambda'(\psi). \tag{2.18}$$

To the curve obtained by eliminating ψ between (2.17) and (2.18) we add such lines of the system (2.15) as correspond to those values of

ψ for which $\lambda'(\psi)$ is discontinuous Now comparing (2.17) and (2.18) with (2.3) and (2.4) respectively, and remembering (2.10), we conclude that the Cartesian equation of Σ is $\sigma = -U(-t)$, if $y = U(x)$ is the equation of the curve Γ. Superimpose the (x, y) plane over the s-plane, the origins and the axes coinciding. Turn the curve Γ through a right angle in the clock-wise direction in its own plane, and revolve it through an angle π round the vertical through the origin, then Γ will come to coincide with Σ.

Further it has been pointed out above that the curve given by $p = \lambda(\psi) + \delta$, $(\delta > 0)$, lies entirely outside the curve given by $p = \lambda(\psi)$ hence the region given by (2.14) lies exterior to Σ.

It is also easy to show that J (s) is a uniform function of s outside Σ. Hence J (s) is an analytic function of s outside Σ, and vanishes at infinity. Σ is entirely enclosed by the circle $|s| = k$.

Now by Cauchy's residue theorem we have

$$\frac{z^\nu}{\nu!} = \frac{1}{2\pi i}\int \frac{e^{uz}}{u^{\nu+1}}\,du, \qquad \ldots \qquad (2.19)$$

the contour of integration being a circle round the origin.

Multiplying both sides of (2.19) by c_ν, and summing under the sign of integration from 0 to ∞, we have

$$\phi(z) = \sum_0^\infty \frac{c_\nu z^\nu}{\nu!} = \frac{1}{2\pi i}\int e^{uz}\left(\sum_0^\infty \frac{c_\nu}{u^{\nu+1}}\right)du, \qquad \ldots \qquad (2.20)$$

$$= \frac{1}{2\pi i}\int e^{uz}\,\mathrm{J}(u)\,du,$$

provided we suppose that the contour of integration is $|u| = k + \delta > k$.

But now it is evident that the contour of integration may be replaced by any closed curve in the u-plane over which J (u) is an analytic function of u.

This proves the first part of Theorem II.

Next, we have to show that the condition (2.2) is *sufficient*. Replace the contour C by a closed curve Σ' parallel to Σ and lying outside it at an arbitrarily small distance ε.

It is evident that $\phi(z)$ is an integral function of z, and we proceed to find a superior limit of $|\phi(\rho e^{\psi i})|$.

Put $u = -(y + ix)$ and $z = \rho e^{i\psi} = \xi + i\eta$, so that the contour of integration, say Γ', in the (x, y) plane has the equation

$$p = \lambda(\theta) + \varepsilon \equiv \lambda_1(\theta),$$

or

$$y = U_1(x), \text{ say,}$$

where θ is the angle which the tangent to the curve makes with the positive x-axis $U'_1(x)$ is monotonic for $|\theta| >$ or $< \pi/2$, and becomes infinite for $|\theta| = \pi/2$, and $U_1''(x) >$ or < 0 according as $|\theta| <$ or $> \pi/2$, except for a finite number of intervals where $U_1''(x) \equiv 0$*.

Now denote by $M(\varepsilon)$ the maximum value of $\left|\frac{J(u)}{2\pi}\right|$ on Γ'. Also $\left|\frac{du}{dx}\right| = |\sec\theta|$, and $\left|\frac{du}{dy}\right| = |\operatorname{cosec}\theta|$.

Therefore

$$|\phi(z)| \leqslant M(\varepsilon) \int e^{(x\eta - U_1(x)\xi)} |\sec\theta| \, dx,$$

$$\leqslant M(\varepsilon) \int e^{(x\eta - U_1(x)\xi)} |\operatorname{cosec}\theta| \, dy \quad \ldots (2.21)$$

It follows that the maximum value of $|\phi(z)|$ depends upon that of exp. $(x\eta - U_1(x)\xi)$ on the path of integration.

In an interval $(x_2 \leqslant x \leqslant x_1)$ in which $U'(x)$ is continuous, and $U''_1(x) \neq 0$,

$F(x) \equiv x\eta - U_1(x)\xi$ is maximum, when

$$U'_1(x) = \tan\theta = \tan\psi; \quad \ldots (2.22)$$

for then

$$F''(x) < 0.$$

This gives $\theta = \psi$, and consequently we have for the corresponding range of values of ψ

$$\left.\begin{aligned} x &= \sin\psi\,\lambda_1(\psi) + \cos\psi\,\lambda'_1(\psi), \\ U_1(x) &= -\cos\psi\,\lambda_1(\psi) + \sin\psi\,\lambda'_1(\psi), \end{aligned}\right\} \quad \ldots (2.23)$$

---

* These properties of $U_1(x)$ follow from those of $U(x)$ discussed in § 2. $\lambda_1(\psi) + \lambda''_1(\psi)$ is always positive so that no discontinuity of $U_1'(x)$ arises.

so that the maximum value of $F(x) = \rho(x \sin\psi - U_1(x)\cos\psi)$

$$= \rho\lambda_1(\psi) = \rho(\lambda(\psi) + \varepsilon).$$

An interval where $U''_1 \equiv 0$ corresponds to a discontinuity of the first kind of $\lambda'_1(\theta)$. If $\theta = \theta_1$ be a point of discontinuity of $\lambda'_1(\theta)$, then x and $U_1(x)$ are discontinuous functions of ψ for a value of ψ, say ψ_1 equal to θ_1. Equations (2.23) hold for $\psi = \psi_1 \pm 0$, but not for $\psi = \psi_1$. Substituting either of $(\psi_1 + 0)$ and $(\psi_1 - 0)$ for ψ in (2.23), we obtain the maximum value of $F(x)$ to be $\rho\lambda_1(\psi_1) = \rho\ \lambda(\psi_1) + \varepsilon$, for $\psi = \psi_1$.

Similarly we can find the maximum value of the expression $(x\eta - y\xi)$ in an interval of y in which $\frac{dx}{dy}$ is continuous, and $\frac{d^2x}{dy^2} \neq 0$.

Hence for all values of ψ in the interval $-\pi \leqslant \psi \leqslant \pi$ we have

$$|\phi(\rho e^{i\psi})| < e^{\rho(\lambda(\psi) + \varepsilon')}, \qquad \ldots \quad (2.24)$$

for every positive ε', and $\rho \geqslant \rho_0(\varepsilon')$.

This completes the proof of our Theorem II.

3. In this section I shall prove the following theorem which is in substance due to Carbson.

THEOREM III. *If* $\limsup\limits_{\rho \to \infty} \dfrac{\log|\phi(\rho^{i\psi})|}{\rho} = \lambda(\psi)$, ... (3.1)

then all such points on Σ *as correspond to continuities of* $\lambda'(\psi)$ *are singular points of* $J(s)$ *and conversely.*

Let us first establish the direct theorem. The Cartesian equation of Σ in the s-plane is

$$\sigma = -U(-t) = V(t), \text{ say} \quad \ldots \quad \ldots \quad (3.2)$$

Now take a point $P(\sigma, t)$ on Σ such that there exists an interval $(t_2 \leqslant t \leqslant t_1)$ in which $V'(t)$ is continuous, and $V''(t) \neq 0$. Then the corresponding value of $\psi >$ or $< \frac{\pi}{2}$, and it follows from the calculations made in § 2 that

$$\lambda(\psi) = V(t)\cos\psi - t\sin\psi \qquad \ldots \quad (3.3)$$

What we have to show now is that if (3.1) is true, then P is a singular point of J (s). Suppose, if possible, that P (σ, t) is a regular point of J (s). Then we can choose $(t_1 - t_2)$ so small that J (s) will be regular outside the chord

$$\sigma = \overline{V}(t), \quad (t_2 \leqslant t \leqslant t_1), \quad \ldots \quad \ldots \quad (3.4)$$

where $\overline{V}(t)(t_1 - t_2) = V(t_2)(t_1 - t) + V(t_1)(t - t_2)$.

Now $\overline{V}(t) >$ or $< V(t)$, for $t_2 < t < t_1$, according as the modulus of the corresponding value of $\psi >$ or $< \pi/2$,* so that the result of substituting $\overline{V}(t)$ for $V(t)$ in (3.3) means a diminuton in the value of $\lambda(\psi)$, which is contrary to the hypothesis†. Hence P is a singular point of J (s). In a similar manner we can establish the result if P lies in an interval of σ in which $\frac{dt}{d\sigma}$ is continuous, and $\frac{d^2t}{d\sigma^2} \neq 0$.

If P is a point of discontinuity of $V'(t)$, or $\frac{dt}{d\sigma}$, then we replace the neighbourhood of P by a small arc lying inside Σ and touching the arcs forming the angle at P.‡

Secondly, we shall show that the condition (3.1) is necessary,

In § 2 we have proved that

$$\limsup_{\rho \to \infty} \frac{\log |\phi(\rho e^{i\psi})|}{\rho} \leqslant \lambda(\psi) \qquad (|\psi| \leqslant \pi) .. \quad (3.5)$$

What we have to show here is that if P is a singular point, then the inequality in (3.5) is inadmissible for the corresponding value of ψ. Suppose, if possible, that it is otherwise. Then we can show, following the proof of the first half of Theorem II, that P lies in the region of regularity of J (s), which is a contradiction.

---

* This follows from the properties of Γ discussed in § 2.

† For (3.1) implies two inequalities, *viz.*.

(*a*) $\qquad |\phi(z)| < e^{\rho(\lambda(\psi) + \varepsilon)}$, for every positive ε and $\rho \geqslant \rho_0(\varepsilon)$,

and (*b*) $\qquad |\phi(z)| > e^{\rho(\lambda(\psi) - \varepsilon)}$, for a corresponding sequence of values of ρ whose limit is infinity.

‡ Here P corresponds to an interval of ψ, say $\psi_1 \leqslant \psi \leqslant \psi_2$, and in order that P be a singular point of J (s) it is both necessary and sufficient that (3.1) be true for $\psi_1 \leqslant \psi \leqslant \psi_2$.

This completes the proof of Theorem III.

4. Now we have to observe that the sign of equality in (3.5) holds for all values of ψ in the range $-\pi \leqslant \psi \leqslant \pi$, except perhaps for a finite number of them for which $\lambda'(\psi)$ is discontinuous. Let $\psi = \psi_1$ be an isolated point of discontinuity of $\lambda(\psi)$. Then the sign of equality in (3.5) will hold for $\psi = \psi_1 \pm 0$, and since $\lambda(\psi)$ is continuous, it holds for $\psi = \psi_1$. *It follows, therefore, that if $J(s)$ is an analytic function of s outside Σ and including infinity, and such points on Σ as correspond to a continuity of $\lambda'(\psi)$ are singular points of it, then the sign of equality holds in (3.5) for $|\psi| \leqslant \pi$.*

Taking note of this and combining Theorems II and III, we have Theorem I.

5. In this section I wish to apply the foregoing considerations to a particular case of a theorem of Hurwitz, which states that*

Given two series of the type

$$f(z) = \sum_{0}^{\infty} \frac{a_m}{z^{m+1}}, \quad F(z) = \sum_{0}^{\infty} \frac{b_m}{z^{m+1}}, \qquad \ldots \quad (5.1)$$

the series

$$\phi(z) = \sum_{0}^{\infty} (a_0 b_m + {}^mC_1\, a_1 b_{m-1} + \ldots + a_m b_0) \frac{1}{z^{m+1}} \qquad \ldots \quad (5.2)$$

has no other singular points than those which one obtains by adding the affixes of a singular point of $f(z)$ to those of a singular point of $F(z)$,

Now in the light of the foregoing theorems we are able to establish the following theorem :—

THEOREM IV. *Suppose the functions*

$$f(z) = \sum_{0}^{\infty} \frac{a_m}{z^{m+1}}, \quad F(z) = \sum_{0}^{\infty} \frac{b_m}{z^{m+1}},$$

are analytic functions of z outside the closed curves Σ_1, and Σ_2, which are derived from the curves given by

$$p = \lambda_1(\psi), \qquad p = \lambda_2(\psi), \qquad (|\psi| \leqslant \pi),$$

* J. Hadamard : '*La Series de Taylor*,' (1901), p. 73

as in § 2, λ_1 and λ_2 satisfying conditions similar to those imposed upon $\lambda\ (\psi)$ in § 1. Suppose further that such points on Σ_1, and Σ_2 as correspond to continuities of $\frac{dp}{d\psi}$ are singular points of $f(z)$ and $F(z)$ respectively. Then

$$\phi(z) = \sum_0^\infty (a_0 b_m + {}^mC_1\, b_{m-1} a_1 + \ldots + b_0 a_m) \frac{1}{z^{m+1}}$$

is an analytic function of z outside the closed curve Σ which is derived from the curve

$$p = \lambda_1(\psi) + \lambda_2(\psi), \qquad (|\psi| \leqslant \pi),$$

in the same manner and such points on Σ as correspond to continuities of $\{\lambda'_1(\psi) + \lambda'_2(\psi)\}$ are singular points of $\phi(z)$.

In the first place we observe that the integral functions

$$u(z) = \sum_0^\infty \frac{a_m z^m}{m!} \quad \text{and} \quad v(z) = \sum_0^\infty \frac{b_m z^m}{m!}$$

satisfy the asymptotic equalities :

$$\lim_{\rho \to \infty} \sup. \frac{\log |u(\rho e^{i\psi})|}{\rho} = \lambda_1(\psi), \qquad \ldots \quad (5.3)$$

and

$$\lim_{\rho \to \infty} \sup, \frac{\log |v(\rho e^{i\psi})|}{\rho} = \lambda_2(\psi), \ (|\psi| \leqslant \pi). \quad (5.4)$$

Secondly. if $w(z) = u(z)\ v(z)$, then the integral function

$$w(z) = \sum_0^\infty (a_0 b_m + {}^mC_1\, a_1 b_{m-1} + \ldots + a_m b_0) \frac{z^m}{m!}$$

and satisfies the asymptotic equality

$$\lim_{\rho \to \infty} \sup. \frac{\log |w(\rho e^{i\psi})|}{\rho} = \lambda_1(\psi) + \lambda_2(\psi), \ \ldots \quad (5.5)$$

for $|\psi| \leqslant \pi$, so that, applying Theorem I, we have the required resul.

THE SYMMETRY OF THE POLO CONIC TRANSFORMATION.

BY S. AUDINARAYANAN.

The locus of a point P whose polar conic with respect to a given cubic curve Γ is in-polar to a given conic S is known to be a conic S′ called *the Polo conic of S with respect to* Γ, by H. S. White. The general notion of polo-conics derives from Hilbert* and has been discussed by H. S. White.† It is known that the relation between a conic and its polo-conic is symmetric; namely, if S′ is the polo-conic of S, then S is the polo-conic of S′ with respect to Γ. This relation is not explicitly stated by White but seems to have been generally known. I have seen no other proof of the relation but the one given by Gabbatt ‡ using the theory of quadratic transformations and projective pencils. I propose to give two proofs of this symmetrical character of the relation, one by synthetic geometry and the other by symbolic calculus.

2. Synthetic Proof.

Given a cubic curve Γ, let the polar-conic of P with respect to it be a pair of lines through P′; then the polar-conic of P′ with respect to Γ is a pair of lines though P. The locus of such points PP′ is the well-known Hessian cubic Γ', of Γ and P, P′ are termed *corresponding points* on Γ'.

The proof of the symmetric relation between a conic and its polo-conic can be immediately derived from the theorem that:

A conic and its polo-conic with respect to a cubic curve Γ intersect the Hessian cubic Γ' of Γ at points which are respectively the corresponding points on the Hessian.

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* "Lettre addresse à M. Hermite." *Liowville* (4) 4. (1888), p. 249.

† *Transactions of the American Mathematical Society*, Vol. 1 (1900), p. 1.

‡ *Proceedings of the Cambridge Philosophical Society*, Vol. 21 (1922), p. 315

To prove this, suppose that a conic S and its polo-conic S′ with respect to Γ intersect the cubic Γ' in points $P_r P'_r$ $(r = 1, 2 \ldots 6)$ respectively. Now P'_r being a point both on S′ and Γ' has a degenerate polar-conic with respect to Γ in-polar to S. But a pair of lines can be in-polar to a conic only if its vertex lies on the conic. Hence the degenerate polar-conic of P'_r has its vertex on S. It must also lie on Γ', being the corresponding point of P'_r. Hence the corresponding point of P'_r is one of the points of intersection P_r of S and Γ. Thus the polo-conic, S′ of S with respect to Γ is the conic through the six corresponding points of the points of intersection of S and Γ'. But the relation between two correspondig points on a cubic is symmetric, each corresponding to the other. Hence the polo-conic of S′ with respect to Γ is the conic S. Thus the reciprocal relation between S and S′ is established.

3. Symbolic Proof.

Let the symbolic equations of the given cubic Γ, and the conic S, in ternary forms be

$$\Gamma \equiv a_x^3 = b_x^3 = c_x^3 = d_x^3 = 0.$$

$$S \equiv u_x^2 = 0.$$

Then the polo-conic S′ of S with respect to Γ is

$$(ab\,u)^2\, a_x b_x = 0 \equiv A_x^2 B_u^2 \text{ (say)} \qquad \ldots \quad (1)$$

and of S′ with respect to Γ is

$$(cd\,A)^2\, c_x d_x\, B_u^2 = 0 \qquad \ldots \quad (2)$$

i.e. $$A_{\overline{cd}}^2\, B_u^2\, c_x d_x = 0.$$

The explicit form $(abu)^2\, a_x b_x$ in (1) is derivable from the abbreviated form $A_x^2 B_u^2$ by the use of the differential operators

$$\frac{1}{4}\left(a\frac{\partial}{\partial A}\right)\left(b\frac{\partial}{\partial A}\right)\left(ab\frac{\partial}{\partial B}\right)$$

where $\left(ab\frac{\partial}{\partial B}\right)$ stands for

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \frac{\partial}{\partial B_1} & \frac{\partial}{\partial B_2} & \frac{\partial}{\partial B_3} \end{vmatrix}$$

Now applying these differential operators to the expression in (2) we have it explicitly as

$$(abu)^2 (cda)(cdb)\, c_x d_x$$

$$= (abu)(cda)(cdb)\, c_x \, [(dab)\, u_x + (dbu)\, a_x - (dau)\, b_x]$$

$$= (abu)(cda)(cdb)\, c_x \, [(dab)\, u_x + (dbu)\, a_x]$$

$$+ (abu)(cda)(cdb)(dbu)\, a_x c_x$$ interchanging a and b in the last term

$$= (abu)(cda)(cdb)\, c_x \, [(dab)\, u_x + (dbu)\, a_x]$$

$$- (adb)(cda)(cbu)(dbu)\, a_x c_x$$

interchanging a and c in the last term

$$= (abu)(cda)(cdb)(dab)\, c_x u_x$$

$$+ (cda)(dbu)\, a_x c_x \, [(cdb)(abu) - (adb)(cbu)]$$

$$= \tfrac{1}{3} (cda)(cdb)(dab)\, u_x \, [(abu)\, c_x - (cbu)\, a_x - (acu)\, b_x]$$

$$+ (cda)(dbu)^2 (cab)\, a_x c_x$$

$$= \tfrac{1}{3} (cda)(cdb)(dab)(cab)\, u_x^2 - (abu)^2 (cdb)(cda)\, c_x d_x$$

interchanging a and d in the last term.

$$\therefore \quad (abu)^2 (cdb)(cda)\, c_x d_x = \tfrac{1}{6} (cda)(cdb)(dab)(cab)\, u_x^2$$

Thus the polo-conic of S′ is S. This establishes the reciprocal relation between a conic and its polo-conic with respect to an order cubic

ON SOME ELLIPTIC-FUNCTION SERIES

By M. Venkatarama Aiyar, M.A., L.T.

1. In this note, an attempt is made to find the values of a few series met with in Elliptic Function theory.

Let $$y = \frac{1}{8}\cot^2\frac{\theta}{2} + \frac{1}{12},$$

$$z = \sum_{n=1}^{\infty} \frac{nx^n}{1-x^n}(1-\cos n\theta),$$

$$S_r = \sum_{n=1}^{\infty} \frac{n^r x^n}{1-x^n},$$

and $$P_r = \sum_{n=1}^{\infty} \frac{n^r x^n}{1-x^n} \times \cos n\theta.$$

Then, $$(y+z)^2 = y^2 + \frac{1}{12}(5\,S_3 + P_3)^* \qquad \ldots \quad (1)$$

Using the power-series for $\frac{\theta}{2}\cot\frac{\theta}{2}$ and for $\cos n\theta$, and writing

$$A_{2n+1} = \frac{B_{n+1}}{4(n+1)} + (-1)^{n+1} S_{2n+1},$$

(1) could be written:

$$\left\{ \frac{1}{2\theta^2} + \sum_{n=1}^{\infty} \frac{\theta^{2n}}{(2n)!}\cdot A_{2n+1} \right\}^2 = \frac{1}{4\theta^4} + \frac{A_3}{3\,!} + \frac{1}{12}\cdot \sum_{n=1}^{\infty} \frac{\theta^{2n-2}}{(2n-2)\,!}\cdot A_{2n+1}.\dagger$$

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* This result is given by the late Mr. S. Ramanujan, F.R.S., in an article 'On Certain Arithmetical Functions' in the *Transactions of the Cambridge Philosophical Society*, Vol. XXII, p. 162, (18).

† Cf. *loc. cit.*, p. 163, (21).

Equating co-efficients of θ^{2n} we have:

$${}_{2n}C_2 \cdot A_3 A_{2n-1} + {}_{2n}C_4 \cdot A_5 A_{2n-3} + \ldots + {}_{2n}C_{2n-2} \cdot A_{2n-1} A_3$$

$$= \frac{A_{2n+3}}{12} \cdot \frac{(2n-2)(2n+5)}{(2n+1)(2n+2)}, ^{*} \quad \ldots \quad (2)$$

for $n >$ or $= 2$.

This gives, for particular values of n,

$$A_7 = 120 \, . \, A_3^2, \; A_9 = \frac{5040}{11} \cdot A_3 A_5, \; A_{11} = \frac{180 \, . \, 120 \, . \, 56}{13} A_3^2 + \frac{180 \, . \, 70}{13} A_5^2$$

and so on.

2. In result (2), if x is so chosen that A_5 vanishes, then it can easily be proved by induction that $A_9, A_{13}, \ldots \ldots$ all vanish and $A_7, A_{11}, \ldots \ldots$ all come out as successive higher powers of A_3 multiplied by certain constants.

3. If we take $x = e^{-2\pi}$, then

$$A_5 = \frac{B_3}{12} - \left[\frac{1^5}{e^{2\pi}-1} + \frac{2^5}{e^{4\pi}-1} + \frac{3^5}{e^{6\pi}-1} + \ldots \ldots \right] = 0.^{\dagger}$$

Hence, $A_9 = A_{13} = \ldots \ldots = 0$, *i.e.*

$$\frac{1^9}{e^{2\pi}-1} + \frac{2^9}{e^{4\pi}-1} + \frac{3^9}{e^{6\pi}-1} + \ldots \ldots = \frac{B_5}{4.5}, \quad \ldots \quad (3)$$

$$\frac{1^{13}}{e^{2\pi}-1} + \frac{2^{13}}{e^{4\pi}-1} + \frac{3^{13}}{e^{6\pi}-1} + \ldots \ldots = \frac{B_7}{4.7}, \quad \ldots \quad (4)$$

etc.

Also, $$A_7 = \frac{B_4}{4.4} + S_7 = 120 \, A_3^2,$$

that is

$$\frac{B_4}{16} + \frac{1^7}{e^{2\pi}-1} + \frac{2^7}{e^{4\pi}-1} + \frac{3^7}{e^{6\pi}-1} + \ldots \ldots$$

$$= 120 \left(\frac{B_2}{4.2} + \frac{1^3}{e^{2\pi}-1} + \frac{2^3}{e^{4\pi}-1} + \frac{3^3}{e^{6\pi}-1} + \ldots \ldots \right)^2 \ldots \quad (5)$$

etc.

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* This result is given by the late Mr. S. Ramanujan, F.R.S., in an article 'On Certain Arithmetical Functions' in the *Transactions of the Cambridge Philosophical Society*, Vol. XXII, p 163, (22).

† *Vide J.I.M.S.*, Vol. XV, p. 153, (2) with $m = 1$.

4. Now, write out in full result (1) above, and in it, change θ into $(\pi+\theta)$, we have

$$\left\{\left(\frac{1}{8}\tan^2\frac{\theta}{2}+\frac{1}{12}\right)+\frac{x}{1-x}(1+\cos\theta)+\frac{2x^2}{1-x^2}(1-\cos 2\theta)+\ldots\right\}$$

$$=\left(\frac{1}{8}\tan^2\frac{\theta}{2}+\frac{1}{12}\right)^2+\frac{1}{12}\left\{\frac{1^3x}{1-x}(5-\cos\theta)+\frac{2^3x^2}{1-x^2}(5+\cos 2\theta)\right.$$

$$\left.+\ldots\ldots\ldots\right\} \qquad (1\text{-}a)$$

Here, change x into $-x$ in both (1) and $(1\cdot a)$, subtract one from the other; use the power-series in θ for the sum and the difference of

$$\frac{1}{8}\cot^2\frac{\theta}{2}+\frac{1}{12} \text{ and } \frac{1}{8}\tan^2\frac{\theta}{12}+\frac{1}{12}.$$

Take $\qquad T_p=\sum_{n=1}^{\infty}\frac{(2n-1)^p\cdot x^{2n-1}}{1+x^{3n-1}}$; $R_p=\sum_{n=1}^{\infty}\frac{n^p\cdot x^{2n}}{1-x^{2n}}$.

Then we get

$$\left\{\frac{1}{2\theta^2}+\left(\frac{1}{12}-2T_1\right)+\frac{\theta^2}{2!}\left(\frac{B_2}{4.2}+R_3\right)2^4+\frac{\theta^4}{4!}\left(\frac{B_3}{4.3}-R_5\right)\right)2^6$$

$$+\ldots\ldots\Big\}$$

$$\times\left\{\frac{1}{2\theta^2}-\left(\frac{1}{12}-2T_1\right)-\frac{\theta^2}{2!}\left(\frac{B^2}{4.2}\cdot\overline{2^3-1}+T_3\right)2\right.$$

$$\left.-\frac{\theta^4}{4!}\left(\frac{B_3}{4.3}\cdot\overline{2^5-1}-T_5\right)2+\ldots\right\}$$

$$=\frac{1}{12}\left\{\frac{3}{\theta^4}-\left(\frac{7}{120}+2T_3\right)-2.\frac{\theta^2}{2!}\left(\frac{B_3}{4.3}\cdot\overline{2^5-1}-T_5\right)\right.$$

$$\left.-2.\frac{\theta^4}{4!}\left(\frac{B^4}{4.4}\overline{2_7-1}+T_7\right)-\ldots\right\} \quad \ldots \quad (6)$$

5. We notice that the constant terms on both sides are equal, in the case when $\frac{1}{12}-2T_1=0$, which occurs when $x=e^{-\pi}$, for it is known that

$$\frac{1}{e^{\pi}+1}+\frac{3}{e^{3\pi}+1}+\frac{5}{e^{5\pi}+1}+\ldots\ldots=\frac{1}{24}.^*$$

* *Vide J.I.M.S*, Vol. XV. p. 154. (5) with $m=0$.

The constant term on the left is

$$\frac{B_2}{16} + 4R_3 - \tfrac{1}{2}\,T_3$$

and that on the right side

$$-\frac{7}{1440} - \frac{1}{6}\,T_3.$$

Their equality leads on reduction to $T_3 = \frac{1}{48} + 12\,R_3$, a result established elsewhere.*

6. Let $$\frac{B_{n+1}}{4\,(n+1)} + (-1)^{n+1}\,.\,R_{2n+1} = C_{2n+1},$$

and $$\frac{B_{n+1}}{4\,(n+1)}\,.\,(2^{2n+1}-1) + (-1)^{n+1}\,.\,T_{2n+1} = D_{2n+1}$$

Then, (6) becomes

$$\left(\frac{1}{2\theta^2} + 2D_1 + \frac{\theta^2}{2\,!}\,.\,C_3\,.\,2^4 + \frac{\theta^4}{4\,!}\,.\,C_5\,.\,2^6 + \ldots\right)$$

$$\times\left(\frac{1}{2\theta^2} - 2D_1 - 2D_3\frac{\theta^2}{2\,!} - 2D_5\,.\,\frac{\theta^4}{4\,!} - \ldots\right)$$

$$= \frac{1}{12}\left\{\frac{3}{\theta^4} - 2\,D_3 - 2\,D_5\,.\,\frac{\theta^2}{2\,!} - 2\,D_7\,.\,\frac{\theta^4}{4\,!} - \ldots\right\}$$

Here, equate co-efficients of like powers of θ. We have

$$\frac{1}{2.3.4}\,.\,C_5\,.\,2^6 - \frac{1}{2.3.4}\,.\,D_5\,.\,2 - 2_5\,D_1C_3 - 2^2\,D_1D_3 = -\frac{1}{6}\,D_5, \qquad (7)$$

$$\frac{1}{2.5.6}\,.\,C_7\,.\,2^8 - \frac{1}{2.5.6}\,.\,D_7\,.\,2 - 2^7\,.\,D_1C_5 - 2^2\,D_1D_5 - \frac{4\,!}{2\,!\;2\,!}\,C_3D_3.2^5$$

$$= -\frac{1}{6}\,D_7, \qquad \ldots \quad (8)$$

etc.

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* *Vide J.I.M.S.*, Vol. XV, p. 161, (20).

7. When $x = e^{-\pi}$, $D_1 = 0$. and

$$C_3 = \frac{B_2}{8} + R_3 = \frac{B_2}{8} + \frac{1}{12}\left(T_3 - \frac{1}{48}\right) = \left(\frac{B_2}{8}.7 + T_3\right)\frac{1}{12} = \frac{D_3}{12}$$

or $$D_3 = 12\ C_3 \quad \ldots \quad \ldots \quad (9)$$

From (7), $$\frac{8\ C_5}{3} = -\frac{1}{12}.D_5$$

or $$D_5 = -32\ C_5 \quad \ldots \quad \ldots \quad (10)$$

But C_{2n+1} when $x = e^{-\pi}$ is identical with A_{2n+1} when $x = e^{-2\pi}$; also, it was shown that, when $x = e^{-2\pi}$, $A_7 = 120\ A_3{}^2$ in (5)

$$\therefore \quad \text{when } x = e^{-\pi},\ C_7 = 120\ C_3{}^2 \quad \ldots \quad (11)$$

Using this in (8), we derive that

$$D_7 = 105 . 2^8 . C_3{}^2 \quad \ldots \quad \ldots \quad (12)$$

and so on.

ON NUMBERS WHICH CONTAIN NO FACTORS OF THE FORM $p\,(kp+1)$

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In a thesis* submitted to the California University, Henry W. Stager discusses the properties of numbers which contain no factors of the form $p\,(kp+1)$, where p is a prime and k is a positive integer. He conjectures that the number of such numbers, (P's as he calls them) and the number of primes within a given limit differ asymptotically by a constant. The object of this paper is to prove that the above conjecture is incorrect and to prove some results regarding the distribution of the P's.

Theorem I. *If $N(x)$ be the number of P's below x, then*

$$N(x) > c \cdot \frac{x}{\log x} \times \log \log x, \qquad (c > 0)$$

Since the number of primes below x is of order $x/\log x$, this result would show that the conjecture is incorrect.

Let p_i be the ith prime. If q be any prime not of the form $tp_i + 1$ and $> p_i$, then $p_i q$ is a P. So given a p_i, the number of such P's below x is the number of primes not of the form $tp_i + 1$ between p_i and x/p_i. Since $q > p_i$, every such P thus associated with p_i is different from those associated with any other prime. Now the number of P's below x associated with p_i is at least equal to

$$\pi\left(\frac{x}{p_i}\right) - \pi_{1,i}\left(\frac{x}{p_i}\right) - i$$

where $\pi(x)$ is the number of primes and $\pi_{i,1}(x)$ the number of primes of the form $tp_i + 1$ below x. Hence if $p_r \leqslant \sqrt{x} < p_{r+1}$

$$N(x) \geqslant \sum_{i=2}^{r} \left\{ \pi\left(\frac{x}{p_i}\right) - \pi_{i,1}\left(\frac{x}{p_i}\right) - i \right\}$$

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* *University of California Publications in Mathematics*, Vol. 1, No. 1, pp. 1—26 (1912).

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$$\sim \sum_{i=2}^{r} \left\{ \frac{x}{p_i \log (x/p_i)} - \frac{1}{\phi (p_i)} \times \frac{x}{p_i \log (x/p_i)} - i \right\}$$

$$\geqslant \sum_{i=2}^{r} \left\{ \frac{p_i - 2}{p_i - 1} \times \frac{1}{p_i} \cdot \frac{x}{\log x} \right\} + O(r^2)$$

$$\geqslant \tfrac{1}{2} \sum_{i=2}^{r} \frac{x}{\log x} \cdot \frac{1}{p_i} + O(r^2)$$

$$\sim \tfrac{1}{2} \frac{x}{\log x} \log \log p_r + O(r^2)$$

$$\therefore \quad N(x) \geqslant \tfrac{1}{2} \frac{x}{\log x} \log \log \sqrt{x} + O(\pi(\sqrt{x}))$$

$$\sim \tfrac{1}{2} \frac{x}{\log x} \log \log x + O\left(\frac{x}{\log^2 x}\right)$$

$$\sim \tfrac{1}{2} \frac{x}{\log x} \cdot \log \log x$$

THEOREM II. $N(x) = o(x)$.

In $\Pi' \left(1 - \frac{1}{c}\right)$ let the accent indicate that for the product of c's, we should take only their least common multiple. Then

$$N(x) = [x] \left[\Pi'_{i,k} \left(1 - \frac{1}{c_{i,k}}\right) \right]$$

where $c_{i,k} = p_i (kp_i + 1)$ and the square bracket indicating that the product is expanded (remembering the meaning of the accent) and the integral part of each term with its proper sign is taken Hence $c_{i,k}$ may have proper limit or run to ∞. (For proof, see the original paper page (6) on-wards).

Let q be any prime, and n a positive quantity very great compared to q. Further let $\Pi \left(1 - \frac{1}{p_t r_{q,t}}\right)$ indicate that $r_{q,t}$ runs through all the primes of the form $qm + t$ not exceeding n.

Then $N(x) \leqslant [x] \left[\prod'_{p_t<q} \prod' \left(1 - \frac{1}{p_t \, r_{q,t}}\right) \right]$

$$= x \prod'_{p_t<q} \prod' \left(1 - \frac{1}{p_t \, r_{q,t}}\right) + O\left(2^{\pi(n)}\right)$$

For different t and s, $p_t \, r_{q,t}$ and $p_s \, r_{q,s}$ are prime to each other.

Hence $N(x) \leqslant x \prod_{p_t<q} \prod' \left(1 - \frac{1}{p_t \, r_{q,t}}\right) + O(2^n)$

$$\prod_{r_{q,t} \leqslant n} \left(1 - \frac{1}{r_{q,t}}\right) \leqslant \frac{A_q}{(\log n)^{1/(q-1)}}$$

$$\therefore \quad \prod' \left(1 - \frac{1}{p_t \, r_{q,t}}\right) = \left(1 - \frac{1}{p_t}\right) + \frac{1}{p_t} \prod_{r_{q,t} < n} \left(1 - \frac{1}{r_{q,t}}\right)$$

$$\leqslant \left(1 - \frac{1}{p_t}\right) + \frac{A_q}{p_t (\log n)^{1/(q-1)}}$$

$$\leqslant \left(1 - \frac{1}{p_t}\right) \left(1 + \frac{A_q}{(\log n)^{1/(q-1)}}\right)$$

$$\therefore \quad N(x) \leqslant x \prod_{p_t<q} \left\{ \left(1 - \frac{1}{p_t}\right) \left(1 + \frac{A_q}{(\log n)^{1/(q-1)}}\right) \right\} + O(2^n)$$

$$\leqslant x \prod_{p_t<q} \left(1 - \frac{1}{p_t}\right) \times e^{A_q \, . \, q/(\log n)^{1/}} + O(2^n)$$

Given δ, for all $q \geqslant q_0 = q(\delta)$

$$\prod_{p_t<q} \left((1 - \frac{1}{p_t}\right) < \delta \qquad \ldots \qquad \ldots \quad (1)$$

Thus fixing q, take n so that $\log n = (A_q \, . \, q)^q$

Then $\qquad eq \, . \, A_q / (\log n)^{1/q} = e \qquad \ldots \qquad \ldots \quad (2$

If $\log x > n$, then

$$2^n = O(x^{\log 2}) \qquad \ldots \qquad \ldots \quad (3)$$

$\therefore$ For all values of $x > x_0 = x(n) = x(q) = x(\delta)$ from (1), (2), (3), we get

$$N(x) \leqslant \delta \, . \, xe + O(x^{\log 2}) = O(\delta \, . \, x)$$

$$\therefore \qquad N(x) = o(x)$$

THEOREM III. *We can find any number of consecutive integers none of which is a P.*

The simultaneous congruence

$$x + 2(r-1) - p_r \equiv 0 \pmod{p_r^2} \qquad (r = 2, 3, \ldots, n) \ldots \quad (4)$$

is solvable. If x_0 is the lowest solution

$$x_0 < (3.5 \ldots \ldots p_n)^2 + p_n = c^2 + p_n \text{ (say)}.$$

Now take $a_0 = x_0 + 4c$ or $x_0 + 5c$ according as x_0 is odd or even, so that a_0 is necessarily odd.

If there is a power of 2 in the interval $(a_0, a_0 + 2n)$, then take $a_1 = a_0 + 2c$

$$\therefore \qquad 2a_0 - (a_1 + 2n) = 2a_0 - (a_0 + 2c + 2n)$$

$$= a_0 - 2c - 2n \geqslant 2c - 2n > 0$$

since $a_0 > 4c$ and $c = (3 \cdot 5 \ldots p_n)^2 > n$

Therefore, if there is a power of 2 in the interval $(a_0, a_0 + 2n)$, since the next power of 2 is $\geqslant 2a_0$, there is no power of 2 in the interval $(a_1, a_1 + 2n)$. Take $b = a_0$, or a_1 so that there is no power of 2 in the interval $(b, b + 2n)$.

Now b is a solution of the congruences (4) since it differs from x_0 by a multiple of c. Hence

$$b + 2r - 2 = p_r(mp_r + 1) \qquad (r = 2, 3 \ldots n).$$

Hence no odd number in the interval $(b, b + 2n)$ is a P, and no even number which is not a power of 2 can be a P. Thus none of the numbers

$$b + 1, b + 2, \ldots, b + 2n$$

is a P. Also $n > k \cdot \dfrac{\log b}{\log \log b}$.

Hence the theorem.

THEOREM IV. *No number which is divisible by p and contains at least $\phi(p)$ other prime factors is a P.*

that is, if

$$n = p^a \cdot p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots p_r^{a_r}$$

where $a \geqslant 1$ and $p, p_1, \ldots p_r$ are different primes and if

$$a_1 + a_2 + \ldots + a_r \geqslant \phi(p),$$

then n is not a P.

To prove this, we require the following two lemmas :—

Lemma 1. If $c, c_2 \ldots, c_r$ are r different residues mod. k, and and s are given positive integers, then the system of numbers

$$c_h + jt \begin{cases} h = 0, 1, 2 \ldots, r \\ j = 0, 1, 2, \ldots, s \end{cases}$$

where $c_0 = 0$, and simultaneous zero values of h and j are excluded, comprise either the zero residue or at least $r + s$ different residues modulus k.

Proof. If no c is congruent to a multiple of t mod k, then the lemma is true; for if $t, 2t, \ldots, st$ are all different residues we get at least $r + s$ different residues or if $t, 2t, \ldots, st$ are not different residues, then $mt \equiv nt$ (mod. k) and hence $(m - n)\,t \equiv 0$ where $0 < m - n < s$ and we get the zero residue.

Next, let h of the c's say $c_1, c_2 \ldots \ldots c_h$ be congruent to multiples of t.

$$\left.\begin{aligned} \text{Then } c_1 &\equiv i_1 t \\ c_2 &\equiv i_2 t \\ \ldots & \quad \ldots \\ c_h &\equiv i_h t \end{aligned}\right\} \text{(mod. } k) \text{ where } 0 \leqslant i < k$$

Since c's are different residues, the i's are different.

Now the system of residues

$$c_g + jt \begin{cases} g = 1. 2, \quad \ldots, h \\ j = 0, 1, 2, \ldots, s \end{cases}$$

is equivalent to the residue system

$$i_g t + jt \begin{cases} g = 1, 2, \ldots, h \\ j = 0, 1, \ldots, s \end{cases} \text{(mod. } k) \text{ where } 0 \leqslant i_g < k \qquad \ldots \quad (1)$$

Let $\qquad (k, t) = d, \quad t = dt', \quad k = dk.'$

Then dividing throughout by d in (1), we get the system of residues

$$i_g t' + jt' \begin{cases} g = 1, \quad \ldots, h \\ j = 0, 1, \ldots, s \end{cases} \text{mod. } k' \qquad \ldots \quad (2)$$

Now reduce i's modulus k' so that

$$i_g \equiv i'_g \text{ (mod. } k'), \ 0 \leqslant i'_g < k'.$$

Then, we get the residues

$$i_g't' + jt' \begin{cases} g = 1, 2, \dots, h \\ j = 0, 1, \dots, s \end{cases} \text{(mod. } k') \quad \dots \quad (3)$$

Also all the i's in (3) are different, for $i'_a = i'_b$ implies $i_a \equiv i_b$ (mod. k') and hence multiplying by t, that $c_a \equiv c_b$ (mod k) which is contrary to hypothesis.

Now without loss of generality we can assume that the suffixes are so chosen that $i'_1 < i'_2 < \dots < i'_h$.

(i) Let $$i'_h + s \geqslant k'.$$

Then since $i'_h < k'$, there is a zero residue mod. k' among

$$i'_h t', \ (i'_h + 1) t', \dots, (i'_h + s) t'.$$

Let it be $(i_h' + f) t'$.

Then

$$(i_h' + f) t' \equiv 0 \text{ (mod. } k')$$

$$\therefore \quad (i_h + f) t' \equiv 0 \text{ (mod. } k')$$

$$(i_h + f) t \equiv 0 \text{ (mod. } k).$$

Hence in the system (1), there is zero residue mod. k.

(ii) Let $$i'_1 = 0$$

Then

$$i_1 \equiv 0 \text{ (mod. } k') ; \quad i_1' \equiv 0 \text{ (mod. } k')$$

$$i_1 t \equiv 0 \text{ (mod. } k)$$

Hence in the system (1), there is a zero residue mod. k.

(iii) Lastly consider the case when

$$i'_h + s < k' \text{ and } i_1' > 0.$$

Now $i'_1 t', (i'_1 + 1) t', \dots, (i'_1 + s) t'$,

$$(i'_2 + s) t', (i'_3 + s) t', \dots, (i'_h + s) t'$$

have all different residues mod. k', for if possible, let

$$(i'_g + u) t' \equiv (i'_f + v) t' \text{ (mod. } k')$$

where, when $g, f > 1$, $u = v = s$ ... ... (a)

and when $g > 1$, $u = s$ and $f = 1$ ... ... (b)

and when $g = f = 1$, $u \neq v$ (c)

Then since $(t', k') = 1$, $0 \leqslant u \leqslant s$.

$$(i_g' + u) \equiv i_f' + v \pmod{k'}$$

(a) $i_g' + s \equiv i_f' + s \quad \therefore \quad i_g' \equiv i_f' \pmod{k'}$.

But $i_g', i_f' < k'$; hence $i_g' = i_f'$ contrary to our supposition.

(b) $i_g' + s \equiv i_f' + v \pmod{k'}, 0 \leqslant v \leqslant s$

$\therefore$ $i_g' + s - i_f' - v \equiv 0 \pmod{k'}$.

But since, $i_g + s < k'$, and $i_g' > i_1'$ and $v \leqslant s$

$$0 < i_f' + s - i_f' - v < k.$$

Hence (b) is also impossible.

(c) $i_1' + u \equiv i_1' + v \pmod{k'}$

$\therefore$ $u \equiv v \pmod{k'}$

Let $u > v$; then $(u - v) t' \equiv 0 \pmod{k'}$. But $0 < u - v < k'$ by our hypothesis and $(t', k') = 1$

Hence this is also impossible. Thus we get, all the numbers in (4) are different residues mod. k'. But $i_g' \equiv i_g$ (mod. k')

$\therefore$ In the system
$$\begin{matrix} (i_1 + u) t' \\ (i_g + s) t' \end{matrix} \quad \begin{cases} u = 0, 1, \ldots, s \\ g = 2, 3, \ldots, h \end{cases} \qquad (5)$$

all the numbers have different residues mod. k'.

Multiplying throughout by d in (5), we get that all the numbers in
$$\begin{matrix} (i_1 + u) t \\ (i_g + s) t \end{matrix} \quad \begin{cases} u = 0, 1, \ldots, s \\ g = 2, 3, \ldots, h \end{cases}$$

have different residues mod. k.

Now in this there are $s + 1 + h - 1 = s + h$ residues. By our assumption, the remaining c's are not congruent to any of these multiples of t and hence they are different from these $s + h$ residues, and by

assumption they themselves are different. Hence in this case we get $r + s\ (= s + h + r - h)$ different residues mod k.

Thus in case (2) either the original system contains a zero residue or comprises at least $r + s$ different residues mod k.

Hence the lemma is completely proved.

Lemma 2. If $a_1 + a_2 + \ldots + a_r \geqslant k$ and $t_1, t_2, \ldots, t_r$ and k are given integers, then the congruence

$$t_1 x_1, + t_2 x_2 + \ldots + t_r x_r \equiv 0 \ (\text{mod. } k) \qquad (5)$$

always possesses a non-simultaneous-zero integral solution in x's satisfying the condition

$$0 \leqslant x_l \leqslant a_l \qquad (l = 1, 2, \ldots, r).$$

We prove the lemma by induction. Assume that by giving to

$$t_1 x_1 + t_2 x_2 + \ldots + t_{h-1} x_{h-1}$$

all possible systems of non-simultaneous-zero values satisfying the above condition, we get either the zero residue or at least

$$a_1 + a_2 + \ldots + a_{h-1} \ (= s \text{ say})$$

different residues mod. k. Now let us consider the next term $t_h x_h$. If we do not get the zero residue, let the different residues already got be $c_1, c_2 \ldots, c_s$ mod. k and put $a_h = u$ and $t_h = b$.

Now in giving all systems of non-simultaneous zero values satisfying the above condition to

$$t_1 x_1 + t_2 x_2 + \ldots + t_{h-1} x_{h-1} + t_h x_h$$

we get at least the values in the system

$$c_i + jb \begin{cases} i = 0, 1, \ldots s \\ j = 0, 1, \ldots u \end{cases}$$

c_0 being equal to 0, and simultaneous zero values of i and j being excluded.

Now by lemma (1), this system contains either the zero residue or at least

$$s + u \ (= a_1 + a_2 + \ldots + a_{h-1} + a_h)$$

different residues mod. k.

To complete the induction, it has to be shown that the lemma is true when $r = 1$. This is obvious for if $t_1 x_1 \equiv 0$ mod. k, $(0 < x_1 \leqslant a_1)$ is not possible, by giving to x_1 the values 1, 2, ... , a_1, we get a_1 different residues mod. k.

Hence when we give to $x_1, x_2 \ldots x_r$ in (5) all except simultaneous zero values satisfying the above condition, we should get either the zero residue or at least $a_1 + a_2 + \ldots + a_r$ different residues mod. k. But since $a_1 + a_2 + \ldots + a_r \geqslant k$ we get all the residues mod. k. Hence the original congruence has got at least one set of non-simultaneous zero roots satisfying the condition. Thus the lemma is proved.

Now we are in a position to prove theorem III.

It is enough if we prove that $m = \frac{n}{p^a}$ has got a factor of the form $kp + 1$, that is, there is a number

$$y = p_1^{x_1} p_2^{x_2} \ldots p_r^{x_r} = tp + 1$$

where $0 \leqslant x_i \leqslant a_i$ $(i = 1, 2, \ldots, r)$ and at least one x is not zero.

Let g be a primitive root of p and

$$p_s = g^{t_s} \quad (s = 1, 2, \ldots, r)$$

Then $y = g^{t_1 x_1 + t_2 x_2 + \ldots + t_r x_r} \equiv 1$ (mod. p) $(0 \leqslant x_s \leqslant a_s)$. and at least one x is not zero. Therefore we should have

$$t_1 x_1 + t_2 x_2 + \ldots + t_r x_r \equiv 0 \quad (\text{mod. } p - 1)$$

with the same conditions.

Since $a_1 + a_2 + \ldots + a_r \geqslant p - 1$, it follows from lemma 2, that there is at least a non-simultaneous zero solution.

Hence the theorem is proved.

In the, original paper, the author gives some theorems which are only particular cases of this, for example, he proves that no multiple of 3 which contains at least 2 other prime factors is a P.

SOME THEOREMS RELATING TO A NET OF PLANE CUBIC CURVES HAVING THREE GIVEN ASYMPTOTES.

BY N. DURAIRAJAN, M.A., B.E.

Notation. The triangle ABC formed by the three fixed asymptotes of the net of plane cubic curves is taken as the triangle of reference. PQR is any line cutting the curve in p, q, r and the sides in P, Q, R. It is known that the mean centre of P, Q, R, coincides with that of p, q, r. We call this point M the *g-point* of the line PQR and the line itself a *g-line* through M. In the following investigation the co-ordinates are areal or Cartesian, according to convenience. In areal co-ordinates, if $lx + my + nz = 0$ be the equation of the line PQR, then the areal co-ordinates of M, its g-point, are:

$$\xi:\eta:\zeta = \left(\frac{n}{n-l}+\frac{m}{m-l}\right) : \left(\frac{n}{n-m}+\frac{l}{l-m}\right) : \left(\frac{m}{m-n}+\frac{l}{l-n}\right)$$

In Cartesian co-ordinates, the equations of the sides are $u_1 = 0$, $u_2 = 0$, $u_3 = 0$

where $$u_r = a_r x + b_r y + c_r$$

and the equation of a variable line is $lx + my + n = 0$. The co-ordinates of its g-point are:

$$\tfrac{1}{3}\sum \frac{mc_1 - nb_1}{lb_1 - ma_1},\ \tfrac{1}{3}\sum \frac{na_1 - lc_1}{lb_1 - ma_1}$$

THEOREM 1. *Two and only two g-lines can in general be drawn through any given point.*

Using rectangular Cartesians, let (h, k) be the co-ordinates of the given point M. The equation of any line through (h, k) is

$$\frac{x-h}{l} = \frac{y-k}{m} = r$$

and in order to be a g-line, the three values of r satisfy $r_1 + r_2 + r_3 = 0$.

$$\therefore\quad \frac{a_1h + b_1k + c_1}{a_1l + b_1m} + \frac{a_2h + b_2k + c_2}{a_2l + b_2m} + \frac{a_3h + b_3k + c_3}{a_3l + b_3m} = 0 \quad \dots \quad (1)$$

This is a quadratic in $l : m$; hence the theorem.

Cor. (i) The two g-lines through a point are perpendicular if the point lies on the radical axis of the circum-circle and polar circle of the triangle.

The lines are perpendicular if (h, k) lies on

$$L = \Sigma (a_2 a_3 + b_2 b_3)(a_1 x + b_1 y + c_1) = 0. \qquad \ldots \quad (2)$$

The lengths of the perpendiculars from A, B, C, on $L = 0$ are proportional to $\left(\frac{a_2 a_3 + b_2 b_3}{a_2 b_3 - a_3 b_2}\right)$ etc., that is are proportional to cot A, etc. The areal equation of the line $L = 0$ referred to the triangle ABC is therefore

$$x \cot A + y \cot B + z \cot C = 0. \qquad \ldots \quad (3)$$

This is the radical axis of the circum-circle and polar-circle of the triangle.*

Cor. (ii) The two, g-lines through a point M conicide if M lies on the maximum inscribed ellipse of the triangle.

The condition for the equality of the two values for l/m in equation (1) gives, on reduction:

$$u_1^2 (a_2 b_3 - a_3 b_2)^2 + \ldots - 2u_2 u_3 (a_3 b_1 - a_1 b_3)(a_1 b_2 - a_2 b_1) - \ldots = 0 \qquad \ldots \quad (5)$$

or

$$\Sigma\, \alpha^2 \sin^2 A - 2\, \Sigma\, \beta\gamma \sin B \sin C = 0 \qquad \ldots \quad (6)$$

where $\alpha = (a_1 x + b_1 y + c_1)/\sqrt{a_1^2 + b_1^2}$, with similar expresssions for β and γ.

$\therefore$ Its equation in areals is

$$S = x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0 \qquad \ldots \quad (7)$$

which represents the maximum inscribed ellipse of the triangle ABC.

Cor. (iii) The g-lines are imaginary if the point M lies within the maximum inscribed ellipse.

This result may be stated otherwise thus: The mean centre of the points of intersection of a real line with the sides of a triangle lies either on or outside the maximum inscribed ellipse.

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* Gallatly's ***Modern Pure Geometry***. 2nd Edition, Art. **21**.

Cor. (iv) The g-lines through a point include a constant angle, if the point lies on a conic of the form $S - kL^2 = 0$, where k is a constant.

2. From the property that the intersections of a line with a curve as well as its intersection with its asymptotes have a common centroid, we deduce the following result:

THEOREM 2. If M be the node of any cubic curve having the sides of ABC for asymptotes, then the g-point of either nodal tangent is M itself, and the two g-lines through M are the nodal tangents of that curve of the net which has a node at M.

Hence applying the results of § 1, we deduce the following:

(i) The locus of the cusps of cuspidal cubics of the net is the maximum inscribed ellipse of the triangle formed by the asymptotes.*

(ii) The locus of the nodes of orthogonal curves of the net is the radical axis of the circum-circle and polar circle of the triangle of asymptotes; and generally the locus of the node of curves, whose nodal tangents cut at a given angle α, is the conic

$$S - \tan^2\alpha\,(x \cot A + y \cot B + z \cot C)^2 = 0. \quad \ldots \quad (8)$$

(iii) The polar conics of a given point with respect to all cubics of the net are conics of equal eccentricity, the species of the conic depends on the position of the point, *i.e.*, whether it lies within, on, or outside the maximum inscribed ellipse. The polar conics are rectangular hyperbolas, when the point lies on $L = 0$

3. THEOREM 3. The envelope of the cuspidal tangents of the cuspidal cubics of the net is a biparabola.†

The g-point of the line $lx + my + nz = 0$ lies on the conic $S = 0$, and its co-ordinates are given by

$$(m - n)(nl + lm - 2mn) : (n - l)(lm + mn - 2nl) : (l - n)(mn + nl - 2lm)$$

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* Hilton: *Plane Algebraic Curves*, p. 32, Ex. 3.

† A biparabola is a curve of class three which touches the line at infinity twice. For a discussion of the properties of the biparabola reference may be made to *J. I. M. S.*, Vol. XVII, p. 77.

This is problem No. 1741 of *Wolstenholme's Problems*. 2nd Edition, 1878, statad slightly differently.

Substituting in $S = 0$ we get the envelope which is apparently a curve of the sixth class, but this turns out to be the square of a curve of class three, whose equation is

$$K_3 \equiv (l + m + n)(mn + nl + lm) - 9\,lmn = 0 \quad \ldots \quad (9)$$

This is a curve which touches the sides and medians of the triangle ABC and the line at infinity at the two imaginary points of intersection of the asymptotes of the maximum inscribed ellipse. This curve is thus seen to be the orthogonal projection of a three-cusped hypocycloid. The conic $S = 0$ has triple contact with the curve.

4. THEOREM 4. The envelope of the nodal tangents of orthogonal curves of the net is a harmonic biparabola.

The g-point of the nodal tangent $lx + my + nz = 0$ lies on

$$x \cot A + y \cot B + z \cot C = 0.$$

The envelope is

$$K'_3 \equiv \cot A\,.(m - n)(nl + lm - 2mn) + \ldots + \ldots = 0 \quad \ldots \quad (10)$$

This is a curve of class three of which line $L = 0$ is the orthoptic locus.

(i) The g-lines of all points on a given line envelope a biparabola which touches the sides of the triangle and the given line

If the given line be $px + qy + rz = 0$, the envelope is

$$\Sigma\, p\,(m - n)(nl + lm - 2mn) = 0 \quad \ldots \quad (11)$$

(ii) The envelope of the nodal tangents of curves, which have a given nodal angle α, is one or other of two biparabolas.

Substituting in equation (8) the co-ordinates $(\xi : \eta : \zeta)$ in terms of (l, m, n), we get the envelope to be

$$K_3^2 - \tan^2\alpha\,.\,K'^2_3 = 0 \quad \ldots \quad \ldots \quad (12)$$

i.e. the envelope consists of the curves

$$K_3 + \tan\alpha\,.\,K'_3 = 0 \text{ and } K_3 - \tan\alpha\,.\,K'_3 = 0 \quad \ldots \quad (13)$$

5. If M be a point on a cubic curve and if a chord LMN be drawn through M cutting it in L, M, N so that M is the middle point of LN, then LN is defined as the *medical chord* through M.

Through any point M, on the cubic, two medial chords can be drawn, for these medial chords are the two g-lines through M. Knowing that the chord LMN intersects the polar conic of M in a point T such that the range (LMNT) is harmonic*, we deduce that the two medial chords through M are parallel to the asymptotes of the polar conic of M (which passes through M and touches the cubic at M).

THEOREM 5. The envelope of the medial chords † of a cubic is a curve of the sixth class, which has the line at infinity as a triple tangent.

The general equation of the cubic is

$$xyz = (x + y + z)^2 (px + qy + rz) \qquad \ldots \quad (14)$$

The point ($\xi : \eta : \zeta$) lies on the curve and hence the envelope is

$$\Pi(-2mn + nl + lm) = 9\,\Pi(m - n) \times$$
$$[p(m - n)(nl + l\cdot n - 2mn) + \ldots + \ldots] \quad \ldots \quad (15)$$

It can be verified that the line (1, 1 : 1) is a triple tangent. The curve touches the sides of the triangle, the line $px + qy + rz = 0$ which is the satellite of the line at infinity, the nodal and inflexional tangents of the given cubic curve.

The following results are easily verified :—

(i) All cubics of the net which pass through M have the same medial chords through M.

(ii) The cubics of the net which have a given inflexion at M, have one or other of the two g-lines as inflexional tangents.

(iii) The cubics of the net touching a given line at F, pass through a fixed point F′ on the line.

(iv) The g-points of all lines through a given point F lie on a cubic with a node at F and whose linear dimensions are $\frac{1}{3}$ of those of that cubic of the net, which has a node at F.

(v) The locus of g-points of a series of parallel lines is a line which touches the maximum inscribed ellipse of the triangle formed by the asymptotes

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* Hilton : *Plane Algebraic Curves*, p. 95.

† This solves Q. 267, *J. I. M. S.*, Vol. III (1911).

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NOTES AND QUESTIONS

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Notes and Questions.

A Theorem on Equimomental Systems.

Introduction.

The object of this note is to arrive at a symmetric treatment of the problem of finding equimomental systems of a triangle, tetrahedron, and generally of a solid simplex in space of any number of dimensions. The usual text-book treatment which bases such equivalence, on a comparison of the moments and products of inertia of the two systems for a set of rectangular axes, is tedious just because it ignores all considerations of symmetry. By the method detailed below such equivalence, in the case of symmetric distributions of mass follow from the equality of the moments of inertia about a single line, the total mass and the centroid being the same for both the systems. This result is then taken advantage of to show that in space of any number of dimensions n, the solid $(n+1)$-hedron or simplex is equimomental with a single mass at the centroid and a number of equal masses at the centroids of the r-dimensional faces.

1. *Systems equimomental with a triangular lamina*

We start with the following

THEOREM. Two systems are equimomental if they have the same mass M, the same centre of inertia, and the same moments of inertia about any three lines in the plane, no two being parallel. ... (1 1)

To prove this we notice, firstly, that since the two systems have the same mass and the same centroid G, they have the same moments of inertia with respect to three lines drawn through G parallel to the given lines, and hence also about three lines parallel to the given lines, through *any* point P in the plane. Now if we mark off along each line through P, a length r such that M/r^2 measures the moment of inertia of the system about that line, the locus of the end-points of these lines is an ellipse with its centre at P, called the "momental ellipse" of the point P. Hence if we have two mass-systems satisfying the conditions of the theorem, their momental ellipses at any point P have the common

centre P, and intersect in three pairs of diametrically opposite points lying on the three lines through P parallel to the given lines. But two conics with six common points must be identical, and this implies that the two systems have the same moments of inertia about every line through P.

Now a system of three masses M/3 at the mid.-points of the sides of a triangle has the same mass and the same centroid G, as the triangular lamina, and to prove the complete equivalence of the two systems, it is enough to show that they are equimomental about three lines in the plane. But here the symmetry of the distribution comes into play, and if it is shown that they are equimomental about one side of the triangle, a like result will hold for the other two sides as well, and the equivalence will have been established. To complete the proof we calculate the moments of inertia of the two systems about the side BC of length a and find respectively :—

$$I_1 = 2 \cdot \frac{M}{3} \cdot \frac{p^2}{4}; \quad I_2 = \frac{2M}{ap} \int_0^p \frac{xa}{p} \cdot (p - x)^2 \, dx = \frac{Mp^2}{6},$$

p being the distance of BC from the opposite vertex.

The moment of inertia $Mp^2/6$ about BC may also be written in the form $Mp^2/12 + \frac{3}{4} Mp^2/9$ showing that the lamina is equimomental with three masses $M/12$ at the vertices and a mass $\frac{3}{4}M$ at the centroid. It is easy to show, in fact that this is the only equimomental system of four masses, of which three are at the vertices.

2. *Systems equimomental with a tetrahedron.*

For ordinary space the theorem corresponding to (I.1) of paragraph 1 takes the shape :

> Two systems are equimomental if they have the same mass, the same centroid and the same moments of inertia about the edges of a tetrahedron. ... ... **(2.1)**

As before we start by proving that two such systems have the same moments of inertia about six lines drawn through any point P parallel to the edges of the tetrahedron. The momental ellipsoids of the two systems for the point P, are two quadrics whose equations referred to axes through P are, say

$$f(x, y, z) = 1; \text{ and } \phi(x, y, z) = 1,$$

where f and ϕ are homogeneous in x, y, z and hence their curve of intersection lies on the *quadric cone* $f(x, y, z) = \phi(x, y, z)$. The lines through P about which the two systems have the same moments of inertia, are the generators of this quadric cone ; in particular the six lines drawn through P parallel to the edges of the tetrahedron must lie on this cone. But the edges of a tetrahedron can never be all parallel to the generators of a quadric cone, proper or degenerate, since they contain four co-planar triads. Hence the two momental ellipsoids contain points other than those on the quadric cone through their intersection, and it follows that the ellipsoids must be identical. Therefore the two systems have the same moments of inertia about all lines through an arbitrary point P, that is, they are equimomental.

It should be noticed that the theorem still holds if the six lines, about which the two systems are to be equimomental are not the actual edges, but are parallel to the edges of a tetrahedron.

To determine systems of point masses equimomental with a tetrahedron $A_1A_2A_3V$, we start with finding its moment of inertia about a line l drawn parallel to an edge A_1A_2 through the centroid g of the face $A_1A_2A_3$. Let the area of the triangle $A_1A_2A_3$ be A, and Ak^2 its M. I. (moment of inertia) about the line l. A section of the tetrahedron by a plane parallel to the base gives a triangle $A'_1A'_2A'_3$ similar to $A_1A_2A_3$, and its centroid g' lies on the line Vg. Denote by l' a line through g' parallel to A_1A_2 and by x and p, the distances of the vertex V from the lines l' and l. Here p is not the perpendicular distance of the vertex V from the plane $A_1A_2A_3$: let this distance be denoted by cp. Since the section $A'_1A'_2A'_3$ is similar to the base, its moment of inertia about l' is $(x/p)^4\, Ak^2$, and hence the moment of inertia of the slice corresponding to the sections x and $x + dx$ about the line l in the base is

$$c\,dx \left(\frac{x}{p}\right)^4 Ak^2 + \left(\frac{x}{p}\right)^2 A\,.\,c\,dx\,(p - x)^2. \qquad \ldots \quad (2.2)$$

Integrating for x between 0 and p, we have the M. I. of the tetrahedron about l given by

$$\frac{Ak^2pc}{5} + \frac{Acp^3}{30} = \frac{3\,Mk^2}{5} + \frac{Mp^2}{10}. \qquad \ldots \quad (2.3)$$

The volume of the tetrahedron is $Apc/3$, and this has been taken also as its mass M.

We shall now prove that

A homogeneous tetrahedron of mass M is equimomental with a system of

four equal masses M/20 at the vertices, and a fifth mass 4 M/5 at the centroid ; ... (2.41)

or

six equal masses M/10 at the middle-points of the six edges, and a seventh 2 M/5 at the centroid ; ... (2,42)

or

four equal masses 9 M/20 at the centroids of the four faces and a fifth —4 M/5 at the centroid of the tetrahedron. ... (2.43)

To prove the first of these results, write (2.3) in the form

$$\frac{3\,Mk^2}{5} + \frac{M}{20}\,p^2 + \frac{4\,M}{5}\cdot\frac{p^2}{16}. \qquad \ldots \quad (2.5)$$

Now we have seen that three masses M / 12 at the vertices of a triangle and a mass 3 M / 4 at the centroid give the same moment of inertia about any line as the lamina itself. It follows that the first term in (2.5) is the M.I. about l of masses each

$$\frac{1}{12} \times \frac{3M}{5} = \frac{M}{20} \text{ at } A_1,\ A_2 \text{ and } A_3,$$

the mass at the centroid being ignored as l passes through it. The second and third terms give the moment of inertia of a mass M / 20 at the vertex V, and a mass 4M / 5 at the centre of gravity of the tetrahedron. These five masses together sum up to M, have the same centroid as the tetrahedron, and have the same moments of inertia about a line parallel to A_1A_2 and therefore also from considerations of symmetry, of a line parallel to *any* edge of the tetrahedron Hence they form a system equimomental with the tetrahedron.

As for the system (2.42), the three masses M / 10 at the middle-points of the base edges A_1A_2, A_2A_3, A_3A_1 are equimomental with the triangular lamina $A_1A_2A_3$ of mass 3 M / 10, and hence their M. I. about l is $3\,Mk^2 / 10$. The other three masses M / 10 are at the middle-points of the sides of a triangle congruent with $A_1A_2A_3$, and hence their M.I. about l is $\frac{3Mk^2}{10} + \frac{3M}{10}\cdot\frac{p^2}{4}$, while that of the mass $\frac{2M}{5}$ at the centroid about l

is $\frac{2M}{5} \cdot \frac{p^2}{16}$. The total M. I. about l is thus given by the expression found in (2.3) and the equivalence of the two systems follows immediately.

The equivalence of the system (2.43) may be proved exactly from similar consderations, but it is not an independent result and may be deduced from the other two. For this purpose, consider a system of masses $4M/10$ at the mid.-points of the edges, $8M/5$ at the centroid, and an additional system of masses $-3M/20$ at the vertices and $-12M/5$ at the centroid. The combined system is equimomental with the tetrahedron $VA_1A_2A_3$ of mass M. Now the three masses $2M/10$ at the middle-points of the sides of a triangle are equimomental with the triangular lamina itself of mass $6M/10$, and hence also with masses $M/20$ at the vertices of the triangle and $9M/20$ at the centroid of the triangle. Treating each of the four faces in this way, it is clear that the masses at the middle-point of the edges neutralise the negative masses $-3M/20$ at the vertices of the tetrahedron, thus leaving the system equivalent to masses $9M/20$ at the centroids of the faces, and $-4M/5$ at the centroid, which proves (2.43).

3. *Extension to space of n-dimensions.*

The arguments and results of the preceding paragraphs admit of direct extension to space of any number of dimensions n.

Here corresponding to the triangle and the tetrahedron, we have the solid $(n+1)$- hedron or simplex, bounded by $(n+1)$ *primes* (flat regions of $(n-1)$ dimensions) which we call its "faces." The intersections of the faces n at a time give the $(n+1)$ "vertices," and $(n-r)$ at a time the ${}_{n+1}C_{r+1}$ "r-dimensional faces" joining these vertices in sets of $(r+1)$ each. The theorem corresponding to (1.1) and (2.1), namely, that

> Two systems of masses are equi-momental if they have the same total mass, the same centroid, and the same moments of inertia about the ${}_{n+1}C_2$ linear edges of a simplex,

follows by reasoning exactly similar to that used in paras 1 and 2, and the impossibility* of the edges of a simplex being all parallel to the generators of a quadric cone.

---

* The assertion of this impossibility is equivalent to the statement that all the intersection points of $(n+2)$ primes in n-dimensional space do not lie on a quadric. Or again, a quadric cone with given vertex is determined uniquely by

By taking sections parallel to the base $A_1 A_2 \ldots A_n$, and with a notation exactly similar to that used in the tetrahedral case, it may be shown that the M. I. of the simplex V $A_1 A_2 \ldots A_n$ about a line l drawn parallel to an edge $A_1 A_2$ through the centroid g of the base is

$$\int_0^p \left(\frac{x}{p}\right)^{n+1} Ak^2 \, c\,dx + \int_0^p \left(\frac{x}{p}\right)^{n-1} (p-x)^2 \, Ac\,dx$$

$$= \frac{n}{n+2} Mk^2 + \frac{2\,Mp^2}{(n+1)(n+2)} \qquad \ldots \quad (3.1)$$

A being the volume of the base simplex, and the volume A pc/n of the whole simplex being taken as its mass M.

It is now easy to show that analogous to results in two and three dimensions we have:

A solid homogeneous simplex in space of n-dimensions is equimomental with

a system of $(n+1)$ equal masses $M/(n+1)(n+2)$ at the vertices and $(n+1)\,M/(n+2)$ at the centroid; ... (3.20)

or

a system of equal masses, each $4M/(n^2-1)(n+2)$ at the mid.-points of the ${}_{n+1}C_2$ linear edges, and the remaining mass at the centroid; ... (3.21)

... ...

or generally

a system of $(n+1)\,C_{r+1}$ equal masses

$$\frac{M\,(r+1)^2}{(n+1)(n+2)\,{}_{n-1}C_r}$$

at the centres of gravity of each of the r dimensional faces, and the remaining mass at the centroid of the simplex; $(r = 0, 1, \ldots n-1)$. ... (3.2r)

For the first system (3.21), taking M. I. about l, the masses at the vertex V and the centroid G each give $M\,p^2/(n+1)(n+2)$ which is

---

$\frac{n(n+1)}{2} - 1$ lines through it, while the number of edges of the simplex is $n(n+1)/2$. Hence a cone can always be drawn whose generators are parallel to all edges but one, say $A_1 A_2$. This cone breaks up in fact into the two primes opposite to A_1 and A_2, and cannot contain also a line parallel to $A_1 A_2$.

the second term in the expression (3.1). Again assuming the theorem (3.21) true for $(n-1)$-dimensional space the masses at $A_1\ A_2\ \ldots\ A_n$ together with a suitable mass at their centroid g (which will have no M. I. about l) is equivalent to a solid $(n-1)$-dimensional simplex $A_1\ A_2\ \ldots\ A_n$ of mass $Mn\ /\ (n+2)$ and the M. I of this about l is precisely the value of the first term.

For the general case (3.2r), let us denote by $M\ f\ (r,\ n)$ the mass at the centroid of each r-dimensional face, and assume the theorem true for all values of r in $(n-1)$-space, and for values up to $(r-1)$ in n-space. It follows from the first of these assumptions that the masses $M\ f\ (r,\ n)$ at the centroids of the r-dimensional faces of the base simplex $A_1\ A_2\ \ldots\ A_n$ together with a suitable mass at g, are equimomental with the solid simplex $A_1 A_2 \ldots A_n$ of mass $M\ f\ (r,\ n)\ /\ f\ (r,\ n-1)$, and hence their M.I. about l is $M\ k^2\ f\ (r,\ n)\ /\ f\ (r,\ n-1)$ The other masses are at the centroids of the $(r-1)$-dimensional faces of a simplex similar to the base simplex, the ratio of the linear dimensions being $r\ /\ (r+1)$, and hence their M. I. about l is

$$\frac{M\ f\ (r,\ n)\ k^2\ r^2}{f\ (r-1,\ n-1)}\ \frac{r^2}{(r+1)^2} + M\ f\ (r,\ n)\ .\ {}_nC_r\ .\ \frac{p^2}{(r+1)^2}\ ;$$

and lastly the M. I of the mass $M' = M\ \{\ 1 - {}_{n+1}C_{r+1}\ .\ f\ (r,\ n)\ \}$ at the centroid G, is $M'\ p^2\ /\ (n+1)^2$. The sum of all these must make up exactly the expression (3.1). On equating the co-efficients of $M\ p^2$ and $M\ k^2$ in both, we obtain the expression for $f\ (r,\ n)$, used in (3.2r) and a relation connecting $f\ (r,\ n)$, $f\ (r,\ n-1)$, and $f\ (r-1,\ n-1)$, but this is found to be identically satisfied by the above value of $f\ (r,\ n)$.

The induction is completed by noticing that the formula (3.2r) gives correct results for $n=2$ (plane) $r=0$, and 1, as we have independently established.

As immediate consequences of the form of $f\ (r,\ n)$ we deduce

Cor. 1. If $n=2r$, the whole mass is distributed among the centroids of the r-dimensional faces, and there is no mass at the centroid, *e.g.* sides of a plane triangle; triangular faces of four dimensional simplex, etc.

Cor. 2. The mass at the centroid is positive or negative according as $n <$ or $> 2r$.

A. NARASINGA RAO.

Generalization of Some Properties of the Pedal Line.

1. In the note under this title in the last issue of this Journal,* I had stated without proof the following :

THEOREM IV.† A transversal cuts the sides BC, CA, AB of a triangle ABC at L, M, N and B′C′L, C′A′M, A′B′N are drawn perpendicular to BC, CA and AB respectively, forming a triangle A′B′C′ directly similar to ABC. *If the circle A′B′C′ is of constant size, then the envelope of LMN consists of two tricusp-hypocycloids.*

I shall indicate a proof of this theorem and incidentally obtain some properties of these hypocycloids.

2. We require the following

Lemma.—P is any point on a circle centre O and radius c. OX is a fixed direction and $\angle XOP = \theta$. PQ is a line drawn through P making an angle ϕ with OP. If

$$\phi = k\theta + \phi_0 \quad \dots \quad \dots \quad (1)$$

where k and ϕ_0 are constants, then PQ envelops, in general, an epi- or hypo-cycloid according as k is positive or negative.

The only exceptional values of k are:

$k = 0$; when PQ envelops a circle concentric with the given circle ;
$k = -\frac{1}{2}$; when PQ passes through a fixed point on given circle ;
$k = -1$; when PQ remains parallel to itself.

In every other case, the envelope is a proper epi- or hypo-cycloid formed by the rolling of a circle of radius $b = c / 2(k + 1)$ on a circle centre O and of radius $a = ck / (k + 1)$. The given circle is the circumscribed or the inscribed circle according as the curve is an epi- or hypo-cycloid.

This simple form of the equation to an epi- or hypo-cycloid as an envelope does not find mention in text-books like Edward's *Differential Calculus* which usually give the pedal (p, r), the tangential-polar (p, ψ) and the intrinsic (s, ψ) equations of these curves.‡

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* *Journal of the Indian Mathematical Society*, Vol, XVIII, 1, pp. 8–11.
† *Ibid*, p. 11.
‡ Cf. Edward's *Differential Calculus*, p. 345, Art, 413,
Basset : *Cubic and Quartic Curves*, pp. 206-207.
Also Question 1384 in this Journal.

I shall give the proof of this lemma in the particular case of $k = -\frac{3}{2}$; other values of k may be dealt with in a similar manner.

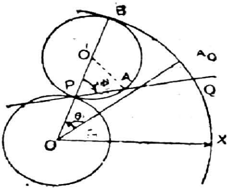

FIG. 1.

Suppose $\theta = \alpha$ when $\phi = 0$, so that $\phi_0 = \frac{3}{2}\alpha$ and equation (1) may be written

$$\phi = -\tfrac{3}{2}(\theta - \alpha) \quad \ldots \quad \ldots \quad (1')$$

Make $\angle XOA_0 = \alpha$. Draw a circle centre O and radius $a = 3c$, cutting OA_0 at A_0. Make $\angle XOP = \theta$ cutting the given circle at P and the second circle at B. Describe the circle on PB as diameter, its centre being O′ and radius $b = \frac{1}{2}(a - c) = c$. Draw through P the line PQ making $\angle BPQ = \phi$ and cutting the circle (O′) at A. Notice that the angles A_0OP ($= \theta - \alpha$) and BPQ ($= \phi$) are of opposite signs in virtue of (1′).

Now $\angle BO'A = 2\phi$

Hence arc $BA = 2c \mid \phi \mid$

$$= 2c \cdot \tfrac{3}{2} \mid \theta - \alpha \mid = 3c \mid \theta - \alpha \mid$$

$$= \text{arc } A_0B$$

Hence A describes the tricusped hypo-cycloid formed by the rolling of the circle, centre O′ and radius c, on the inside of the circle centre O and radius $3c$; and AP, being perpendicular to the line joining A to the point of contact B of the fixed and the rolling circles, is tangent to this hypo-cycloid. The cuspidal tangents are easily seen to be the lines through O for which

$$\theta = \alpha,\ \alpha + \frac{2\pi}{3},\ \alpha + \frac{4\pi}{3}.$$

3. Reverting to the last note, it has been shown that, given the circle A′B′C′ cutting orthogonally* the circum-circle of ABC at O and P

* *J. I. M. S.*, Vol XVIII, 1, p 9, Theorem II.

two positions of the triangle $A'B'C'$ are possible, *viz.*, one which has with ABC, O for centre of similitude and P for centre of perspective and the other having P for centre of similitude and O for centre of perspective.* If the direction of rotation from SO to SP is positive and $\angle OSP = 2\beta$, we shall refer to these positions of the triangle $A'B'C'$ as the $+\beta$-triangle and the $-\beta$-triangle. There are two corresponding positions of the transversal LMN, the $+\beta$-transversal and the $-\beta$-transversal. If the circle $A'B'C'$ is of constant size then β is constant.

4. I shall first consider the $+\beta$ case. H, H' are the orthocentres of ABC and $A'B'C'$; U is the nine-points-centre and S is the circum-centre of ABC. V, V', V_1 are the middle-points of HO, HP and HH', so that the pedal line VW of O with respect to ABC passes through V, the pedal line V'W' of P passes through V' and LMN (V_1W_1) passes

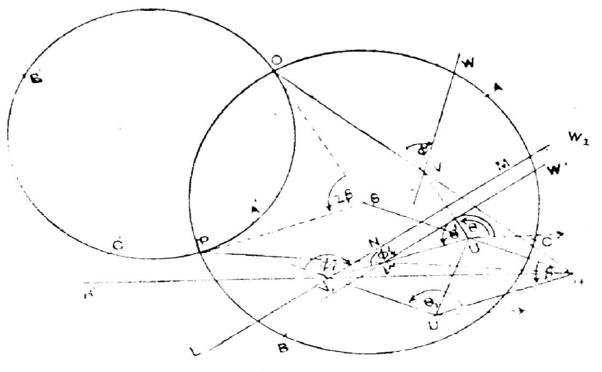

FIG. 2.

through V_1†. Since O is the centre of similitude of the triangles ABC and $A'B'C'$, the triangle formed by O and any pair of corresponding points of ABC, $A'B'C'$ is of given species. Thus $\angle H'OH = 90°$, and $\angle OHH' = +\beta$. It is thus seen that the locus of V' is obtained from the locus of V (the nine-points-circle of ABC) by turning it about H through $+\beta$ and extending it sec β times. Take U_1 on the perpendicular at U

* *J.I.M.S.*, Vol XVIII, 1, p *11*, Art. 7.

† *Ibid.* p. *10*, Theorem III.

to UH making $\angle UHU_1 = +\beta$. Then the locus of V_1 is the circle centre U_1 and radius $\frac{1}{2}$ R sec β, where R is the circum-radius of ABC.

Let θ, θ', θ_1 denote the angles that UV, UV′ and U_1V_1 make with a fixed axis in the plane; ϕ, ϕ', ϕ_1 the angles that VW, V′W′ and V_1W_1 (same as LMN) make with UV, UV′ and U_1V_1 respectively.

Firstly, since $\angle VUV' = \angle OSP = 2\beta$, we have

$$\theta' = \theta + 2\beta \quad \ldots \quad \ldots \quad (2)$$

Then since the pedal lines of O and P touch the same tricusped hypocycloid (the Steiner tricusp of ABC), we have from (1′),

$$\phi = -\tfrac{3}{2}(\theta - \alpha);\ \phi' = -\tfrac{3}{2}(\theta' - \alpha) \quad \ldots \quad (3)$$

the angle α specifying the direction of one of the cuspidal tangents Also since U_1V_1 is obtained by turning UV through an angle $+\beta$,

$$\theta_1 = \theta + \beta \quad \ldots \quad \ldots \quad (4)$$

Lastly, since V_1W_1 (*i.e.* LMN) is parallel to V′W′ the pedal line of P*

$$\theta_1 + \phi_1 = \theta' + \phi' \quad \ldots \quad \ldots \quad (5)$$

each expressing the angle that the parallel lines make with the axis of reference.

From equations (2)—(5), we get

$$\phi_1 = -\frac{3}{2}\left(\theta_1 - \alpha + \frac{\beta}{3}\right).$$

Thus, by the lemma above, the envelope of LMN is a tricusped hypo-cycloid having for its in-circle the circle centre U_1, radius $\frac{1}{2}$R sec β and having each of its cuspidal tangents making an angle $-\frac{1}{3}\beta$ with a cuspidal tangent of the Steiner tricusp.

5. In a like manner, we can deal with the $-\beta$-transversal, *i.e.* the transversal obtained by taking P as the centre of similitude and O as centre of perspective. In this case the envelope will be another tricusped hypo-cycloid having for its in-circle the circle, of radius $\frac{1}{2}$R sec β, having its centre at U_2 where HU_2 = HU sec β and $\angle UHU_2 = -\beta$. Thus we arrive at the following

THEOREM V. *The two tricusped hypo-cycloids are equal in size and have their centres at equal distances on either side of the nine-points-centre on the perpendicular at this point to the Euler line.*

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* *Ibid.* p. 10, art. 4.

6. Another important property of these hypo-cycloids for all values of β is contained in the following :

THEOREM VI. *The hypo-cycloids which are the envelopes of the β-transversals are all inscribed to the given triangle ABC.*

For it is easily seen that each side of ABC is a β-transversal whatever be the value of β.

7. Converse, in his paper* cited in my last note, obtains in addition to the results of Theorems IV, V and VI above, many other interesting properties of these hypo-cycloids [which, after Professor Morley, he refers to as *deltoids*], *e.g.* he obtains the locus of the cusps of these deltoids; he shows that each pair of equal deltoids is the envelope of asymptotes of conics of given eccentricity circumscribing ABC; and so on. He employs, however, the analytical method, using the *circular* co-ordinates of Morley. Also he regards the line LMN not as the axis of perspective of the two similar triangles ABC and A'B'C' but as the Poncelet-line of O — the line containing the points obtained by projecting with a constant angle θ the point O (lying on the circum-circle of ABC) on to the sides of ABC. This line is now familiar as the θ-pedal of O. The line LMN of figure (2) is in fact the θ-pedal of O for the value $\theta = 90^\circ - \beta$. The result of Theorem IV may, after Converse, be stated thus :—

The envelope of the θ-pedals of points on the circum-circle of a triangle for a given value of θ is a deltoid.

G. A. SRINIVASAN.

On Sawayama's Extension of Salmon's Theorem.

In elementary geometry, we have the following well-known property of a circle, known as Salmon's Theorem :—

If A, B are two points in the plane of a circle centre O and if AP, BQ are the perpendiculars from each of the points A, B to the polar of the other, then

$$OA \cdot BQ = OB \cdot AP.$$

* *Annals of Mathematics*, Series II, Vol 5, (1904), pp 105—139.

My attention has been drawn to a much earlier paper : "An Extension of Steiner's Problem," by P. A. Macmahon in the *Messenger of Mathematics*, Vol. XII (1883), pp. 138—41, in which the results of Theorem IV, V and VI are obtained, by analytical methods

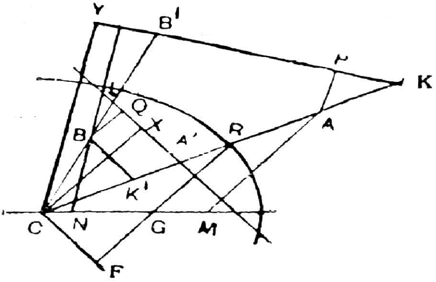
In the *Tohoku Mathematical Journal*, Vol. 22 (1923), pp. 77-78, Y. Sawayama has extended the theorem to a conic in the following manner :—

Take any two points A, B and their polars with respect to a conic S. Let the perpendiculars drawn from each of A and B to its own polar meet an axis of the conic in the points M and N respectively; and let the perpendiculars drawn from each of A and B to the polar of the other point be AP and BQ. Then

$$AM \, . \, BQ = BN \, . \, AP$$

each segment of line having the proper sign according to the rule of analytic geometry.

The proof given by Sawayama is analytical. The following is a geometrical proof based on some well-known results in geometrical conics :—



Let XA′, YB′ be the polars of A, B with respect to the conic; let CA, CB meet XA′ in A′, L and YB′ in K, B′. Let CX, CY be the perpendiculars from C to the polars of A and B respectively. Let CA cut the conic at R; draw the normal at R cutting the axis considered at G and the diameter conjugate to CR at F. Also draw BK′ parallel to XA′, cutting CA in K′.

Now, since XA′ is parallel to the tangent at R, AM, RG and CX are parallel; so from similar triangles we have

$$\frac{AM}{RG} = \frac{CA}{CR} \text{ and } \frac{CX}{RF} = \frac{CA'}{CR}.$$

Multiplying, $$\frac{AM \, . \, CX}{RG \, . \, RF} = \frac{CA \, . \, CA'}{CR^2}$$

But it is well-known that*

$$CA.CA' = CR^2$$

and $$RG.RF = b^2$$

where b represents the semi-axis conjugate to the axis CNM. Hence

$$AM.CX = b^2$$

Similarly $$BN.CY = b^2,$$

so that $$AM.CX = BN.CY \quad \ldots \quad \ldots \quad (1)$$

Again, it is easily seen that BK′ is the polar of K. So

$$CK.CK' = CA.CA' \; (= CR^2.)$$

$$\therefore \quad \frac{CK}{CA} = \frac{CA'}{CK'} = \frac{CL}{CB}.$$

Hence

$$\frac{CK}{AK} = \frac{CL}{BL}$$

Therefore from similar triangles,

$$\frac{CY}{AP} = \frac{CX}{BQ} \quad \ldots \quad \ldots \quad (2)$$

From (1) and (2) we obtain

$$AM.BQ = BN.AP.$$

A. A. KRISHNASWAMI AIYANGAR.

[The above proof may be modified as follows, in the case of the parabola:—

As before, let XA′, YB′ be the polars of A and B. Let the diameters through A, B cut XA′ at A′, L and YB′ at K, B′. The perpendiculars from A, B to the polars of B, A respectively meet the axis at M and N respectively. Let the parallel through B to XA′ meet AA′ at K′. Let AD, BE be the perpendiculars from A, B to the axis. The diameter AA′ cuts the parabola at R.

The triangles ADM, BQL are similar; for the angles at D and Q are right-angles and $\angle DAM = \angle BLQ$, AD, AM being respectively perpendicular to BL, LQ. Hence

$$AM.BQ = DM.BL$$

Similarly $$BN.AP = EN.AK.$$

Now it is easily shown that DM = EN = semi-latus rectum

and $$BL = K'A' = AK.$$

Hence $$AM.BQ = BN.AP.$$

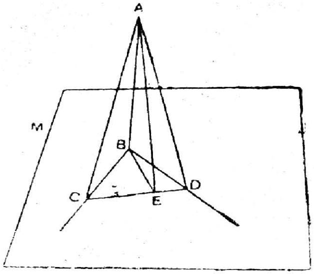
G. A. S.]

* See Smith's *Geometrical Conics*, p. 101, Prop. XV and p. 107, Prop. XIX.

A Theorem in Solid Geometry.

Euclid's proof of Prop. 4, Book XI is notorious for its length and hence simplified proofs have been given by Cauchy, Legendre and others.* The following new proof appears to be interesting and is probably not noticed before.

Euclid : Book XI, Prop. 4. *If a straight line be set up at right angles to two straight lines, which cut one another, at their common point of section, it will also be at right angles to the plane through them.*



Let AB be perpendicular to two straight lines BC, BD in the plane MN at their point of intersection B.

Make BC = BD and let BE be any line in the plane MN cutting CD at E. Join AC, AD, AE.

It is evident that the right-angled triangles ABC, ABD are congruent, so that AC = AD.

From a well-known property of an isosceles triangle, we have, since triangles BCD, ACD are isosceles,

$$BC^2 - BE^2 = CE.ED = AC^2 - AE^2$$
$$\therefore \quad AE^2 = BE^2 + AC^2 - BC^2$$
$$= BE^2 + AB^2$$

whence ABE is a right angle.

Therefore AB is perpendicular to BE, any straight line in the plane MN and hence is perpendicular to the plane MN.

A. A. KRISHNASWAMI AIYANGAR.

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* *Vide* Heath (T.L.): *The Thirteen Books of Euclid's Elements.* Vol III, p. 280.

The Ramanujan Prize.

The following are the terms of endowment of the Ramanujan Prize accepted by the Syndicate of the Madras University and reported to the Senate at its last annual meeting :—

The prize was founded by the subscribers to the Memorial Fund, raised in honour of the great mathematician, S. Ramanujan, B.A. (Cantab), the first Indian F.R.S., who was admitted to that Fellowship at the early age of thirty years.

The endowment at present consists of Rs. 8,800 invested in $3\frac{1}{2}$ p. c. Government Securities; the subscribers have wished to apply the interest to the award of a prize, for mathematical study and research, once in two years, the value of the prize to be not less than Rs. 500.

The prize is to be awarded by the Syndicate to the best essay or thesis written on any branch of mathematics. The prize is to be open to any one born or domiciled in India. The essay or thesis must be the result of the personal investigations of the author and must contain clear evidence of independent and original research.

The Syndicate shall in January of every alternate year call for essays and theses and the prize shall, if possible, be awarded in the month of December of that year. The Syndicate may appoint two or more persons to adjudicate on the essays and theses received. The accepted thesis may be published by the University.

The Syndicate may decline to award the prize on the ground of insufficient merit in the theses received. If no award of the prize be made in any year, the amount of the prize shall be added to the corpus of the fund.

The Syndicate shall have power to make such changes in the conditions of award as new circumstances may render desirable, provided that the prize is associated with the name of Ramanujan and is given for the encouragement of research in Mathematics to one born or domiciled in India.

Solutions.

Question 1019.

(HEMRAJ):—If $p+4$ is a prime, show that

$$B_1 - B_3 + B_5 - + \dots \dots + (-1)^{\frac{1}{2}(p-1)} B_p = \frac{m}{n} - \frac{3}{2}$$

where $|m|$ and n are integers prime to each other and $|m|$ is a multiple of $(p+4)$.

Solution by S. D. Chowla.

Let us write $\qquad \frac{a}{b} \equiv \frac{c}{d} \pmod{p}$

if the numerator of the fraction $\frac{ad-bc}{bd}$, in its lowest terms, is divisible by p.

$B_1, B_2, B_3, \dots \dots$ are the successive Bernoullian numbers (called $B_1, B_3, B_5, \dots \dots$ by the proposer of this Question)

and $\qquad S_n(p) = 1^n + 2^n + \dots \dots + p^n.$

From Bernoulli's formula proved in books on Higher Algebra,* we have

$$S_{2n-1}(a) = \frac{a^{2n}}{2n} + \frac{a^{2n-1}}{2} + \frac{(2n-1)}{2!} a^{2n-2} B_1$$

$$- \frac{(2n-1)(2n-2)(2n-3)}{4!} a^{2n-4} B_2$$

$$+ - \dots + (-1)^{n-2} a^2 (2n-1) B_{n-1}.$$

Now let $r = 2n+1$ be a prime; then since

$$^{2n}C_{2s} \equiv 1 \pmod{r} \text{ when } s = 1, 2, \dots \dots (n-1),$$

and by von Staudt's fundamental theorem† the denominators of $B_1, B_2, \dots \dots B_{n-1}$ do not contain $r(=2n+1)$, we obtain

* *Vide* Chrystal: *Text-Book of Algebra*. Part II, p 209.
Niels Nielsen: *Traite des Nombres de Bernoulli*. p. 142. formula (22).
† Nielsen *l. c.* p. 244.

$$2n\, S_{2n-1}(a) - a^{2n} - na^{2n-1}$$

$$\equiv \{ a^{2n-2} B_1 - a^{2n-4} B_2 + - \dots + (-1)^{n-2} B_{n-1} a^2 \} \text{ (mod. } r) \quad \dots \quad (1)$$

Also, by Fermat's theorem,* if a is prime to r

$$a^{r-1} \equiv 1 \text{ (mod. } r).$$

it follows from (1) that

$$-\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots \dots + \frac{1}{a}\right) - 1 - \frac{r-1}{2a}$$

$$\equiv \left\{ \frac{B_1}{a^2} - \frac{B_2}{a^4} + - \dots \dots + (-1)^{n-2} \frac{B_{n-1}}{a^{2n-2}} \right\} \text{ (mod. } r)$$

whenever a is prime to r.

Hence

$$\sum_{s=1}^{s=n-1} \frac{(-1)^{s-1} B_s}{a^{2s}} \equiv \left\{ -\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots \dots + \frac{1}{a}\right) - 1 + \frac{1}{2a} \right\}$$

$$\text{(mod. } r) \quad \dots \quad (2)$$

if a is prime to r.

If we put $a = 1$ in this and write $r = p + 4$, we get the result of the question.

Both formula (2) and the formula given in the question are found in Niels Nielsen's *Traite des Nombres de Bernoulli* { p. 280, formulae (4) and (5) } .

Question 1143.

(K. Ananda Rao):—If (i) $\lambda(n) = (-1)^s$ where s is the number of prime factors of n (each prime factor being counted according to its degree of multiplicity), (ii) the function $\phi(x)$ satisfies certain conditions, and (iii) $\psi(x) = \sum_{1}^{\infty} \lambda(n)\, \phi(nx)$, then show that $\phi(x) = \sum_{q} \psi(qx)$, the latter summation being taken over all the integers, q, which have no repeated factors.

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* Chrystal, p. 522.

Solution by S. D. Chowla.

Let

$$\psi(x) = \sum_{n=1}^{\infty} f(n)\, \phi(nx) \quad \ldots \quad \ldots \quad (1)$$

then it is obvious that, under certain conditions, we have

$$\sum_{n=1}^{\infty} g(n)\, \psi(nx) = \sum_{n=1}^{\infty} \mathrm{H}(n)\, \phi(nx) \quad \ldots \quad (2)$$

where

$$\mathrm{H}(n) = \sum_{d/n} f(d)\, g\left(\frac{n}{d}\right), \quad \ldots \quad \ldots \quad (3)$$

and d/n means that d is a divisor of n.

Now in the special case when $f(n) = \lambda(n)$ and $g(n) = \mu^2(n)$ where $\mu(n)$ is the well-known Möbius function, $\mathrm{H}(n)$ becomes the co-efficient of n^{-s} in the product of the Dirichlet series $\sum_1^{\infty} \lambda(n)\, n^{-s}$ and $\sum_1^{\infty} \mu^2(n)\, n^{-s}$.

Since

$$\sum_1^{\infty} \lambda(n)\, n^{-s} = \frac{\zeta(2s)}{\zeta(s)},$$

and

$$\sum_1^{\infty} \mu^2(n)\, n^{-s} = \frac{\zeta(s)}{\zeta(2s)} = \sum_q q^{-s},$$

where

$$\zeta(s) = \sum_1^{\infty} n^{-s} \text{ and } \mathrm{R}(s) > 1,$$

we deduce that in this case $\mathrm{H}(1) = 1$ and $\mathrm{H}(n) = 0$ if n is an integer greater than unity. Hence from (1), (2) and (3) we get:

If

$$\psi(x) = \sum_1^{\infty} \lambda(n)\, \phi(nx)$$

then

$$\sum_1^{\infty} \mu^2(n)\, \psi(nx) = \phi(x)$$

Since $\sum_1^{\infty} \mu^2(n)\, \psi(nx) = \sum_q \psi(qx)$, we have obtained the result of the question. As an example we may take $\phi(x) = e^{-x}$.

Question 1348.

(S. D. CHOWLA):—Prove that

(i) $$\frac{e^{-\pi}}{1+e^{-\pi}}+\frac{3e^{-3\pi}}{1+e^{-3\pi}}+\frac{5e^{-5\pi}}{1+e^{-5\pi}}+\ldots\ldots=\frac{1}{24}.$$

(ii) $$\frac{e^{-2\pi}}{1-e^{-2\pi}}+\frac{2^5\,.\,e^{-4\pi}}{1-e^{-4\pi}}+\frac{3^5\,.\,e^{-6\pi}}{1-e^{-6\pi}}+\ldots\ldots=\frac{1}{504}.$$

Remarks by S. D. Chowla.*

The results are got by putting $k = k' = \frac{1}{\sqrt{2}}$ in†

$$1-24\left(\frac{q}{1+q}+\frac{3q^3}{1+q^3}+\frac{5q^5}{1+q^5}+\ldots\right)=\left(\frac{2k}{\pi}\right)^2\left(1-2k^2\right)$$

and

$$1-504\sum_{r=1}^{\infty}\frac{r^5\,.\,q^{2r}}{1-q^{2r}}=\left(\frac{2k}{\pi}\right)^6\left(1-2k^2\right)\left(1+\tfrac{1}{2}\,k^2k'^2\right)$$

where as usual $q = e^{-\pi k'/k}$.

Question 1376.

(V. RAMASWAMI AIYAR):—(i) If a three-cusped hypocycloid touches the sides of a triangle ABC, show that the *third* tangents to the tricusp drawn from A, B, C make equal angles with the opposite sides.

(ii) If the tricusp touch the sides BC, CA, AB in P, Q, R, prove that

$$AB\,.\,AR + AC\,.\,AQ = BC^2$$
$$BC\,.\,BP + BA\,.\,BR = CA^2$$
$$CA\,.\,CQ + CB\,.\,CP = AB^2.$$

Solution by M. V. Seshadri. Also partial solutions by B. Ramamurti and G. V. Krishnaswamy.

(i) The parametric equation to a tri-cusped hypocycloid is

$$x = 2a\cos\theta + a\cos 2\theta$$
$$y = 2a\sin\theta - a\sin 2\theta.$$

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* *Vide* also *J.I.M.S.*, Vol. XVI, p. *172.*

† Halphen: *Fonctions Elliptiques.*

The tangent at 'θ' is

$$x \sin\frac{\theta}{2} + y\cos\frac{\theta}{2} = a\sin\frac{3\theta}{2}.$$

Let BC, CA, AB be tangents at θ, ϕ, ψ respectively.

The tangent at 'λ' passes through (h, k) if

$$h\sin\frac{\lambda}{2} + k\cos\frac{\lambda}{2} = a\sin\frac{3\lambda}{2}$$

i e., if $$t^3(h+a) + t^2 k + t(h-3a) + k = 0$$

where $$t = \tan\frac{\lambda}{2}.$$

So, if $\lambda_1, \lambda_2, \lambda_3$ be the parameters of the three points of contact of the tangents from a point (h, k),

$$\tan\frac{\lambda_1}{2} + \tan\frac{\lambda_2}{2} + \tan\frac{\lambda_3}{2} = \tan\frac{\lambda_1}{2}\tan\frac{\lambda_2}{2}\tan\frac{\lambda_3}{2}$$

i.e., $$\lambda_1 + \lambda_2 + \lambda_3 = 2n\pi,$$

where n is any integer.

The points of contact of two of the tangents from A are ϕ and ψ. Let the third be α; then

$$\phi + \psi + \alpha = 2n\pi.$$

The gradient of the third tangent from A is $-\tan\frac{\alpha}{2}$

and that of BC is $-\tan\frac{\theta}{2}$

$\therefore$ the angle between BC, and the third tangent from A is $\frac{\theta - \alpha}{2}$

$= \frac{\theta + \phi + \psi - 2n\pi}{2}$ which is therefore a constant, when we replace A and BC, by B and CA or C and AB respectively.

(ii) The equations to CA and AB are

$$x\sin\frac{\phi}{2} + y\cos\frac{\phi}{2} = a\sin\frac{3\phi}{2} \quad \ldots \quad \ldots \quad (1)$$

and $$x\sin\frac{\psi}{2} + y\cos\frac{\psi}{2} = a\sin\frac{3\psi}{2} \quad \ldots \quad \ldots \quad (2)$$

Hence the co-ordinates of A are, on solving (1) and (2) and simplifying

$$x_1 = a\,(\cos\phi + \cos\psi + \cos\overline{\phi + \psi})$$
$$y_1 = -\,a\,(\sin\phi + \sin\psi - \sin\overline{\phi + \psi})$$

Similar results hold for (x_2, y_2) and $(x_3\ y_3)$, co-ordinates of B and C respectively.

The following results are easily calculated :—

$$BC = 4a \sin\frac{\phi - \psi}{2} \sin\frac{\theta + \phi + \psi}{2}$$

$$CA = 4a \sin\frac{\theta - \psi}{2} \sin\frac{\theta + \phi + \psi}{2}$$

$$AB = 4a \sin\frac{\phi - \theta}{2} \sin\frac{\theta + \phi + \psi}{2};$$

supposing $\phi > \theta > \psi$

Also $$AQ = 4a \sin\frac{\phi - \psi}{2} \sin\frac{2\phi + \psi}{2}$$

$$AR = 4a \sin\frac{\phi - \psi}{2} \sin\frac{2\psi + \phi}{2}$$

and so on.

Thus AQ . AC + AR . AB

$$= 16a^2 \sin\frac{\phi - \psi}{2} \sin\frac{\theta + \phi + \psi}{2}$$

$$\left[\sin\frac{2\phi + \psi}{2} \sin\frac{\theta - \psi}{2} + \sin\frac{2\psi + \phi}{2} \sin\frac{\phi - \theta}{2}\right]$$

$$= 8a^2 \sin\frac{\phi - \psi}{2} \sin\frac{\theta + \phi + \psi}{2}$$

$$\left[\cos\frac{\theta - 2\psi - 2\phi}{2} - \cos\frac{\theta + 2\phi}{2} + \cos\frac{2\psi + \theta}{2} - \cos\frac{2\psi + 2\phi - \theta}{2}\right]$$

$$= 16a^2 \sin^2\frac{\phi - \psi}{2} \sin^2\frac{\theta + \phi + \psi}{2}$$

$= BC^2$.

The other results are similarly proved.

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1538. (T. VIJAYARAGHAVAN):—Suppose that

$$\sqrt[3]{2} = [x_1, x_2, \ldots x_n \ldots] = x_1 + \frac{1}{x_2 +}\ \frac{1}{x_3 +} \cdots$$

where the x's are positive integers. Further, let

$$\theta_n = [x_{n+1}, x_{n+2}, \ldots].$$

Prove that

$$\theta_n = \frac{a_{n+1} + b_{n+1} \cdot 2^{\frac{2}{3}} + c_{n+1} \cdot 2^{\frac{1}{3}}}{d_{n+1}}$$

where a's, b's, c's and d's are integers between which the following relations hold:

$$a_{n+1} = x_n (x_n d_n - 2 a_n) - x_{n-1} d_{n-1} + a_{n-1},$$

$$b_{n+1} = x_n b_n + b_{n-1},$$

$$c_{n+1} = x_n c_n + c_{n-1},$$

$$d_{n+1} = (a^2_{n+1} - 2 b_{n+1} c_{n+1}) / (-x_n d_n + a_n),$$

$$d_{n+1} = (-1)^n (c_{n+1}{}^3 - 2b^3_{n+1}).$$

Also, express d_{n+1} linearly in terms of the preceding d's, etc., and powers of x's.

1539. (R VAIDYANATHASWAMY):—ϕ_r (N) is the Jordan function equal to the number of sets of r numbers less than N, whose greatest common divisor is prime to N. τ and σ_r are the usual functions representing the number of divisors and the sum of the rth powers of the divisors. Show that,

(1) $$\frac{\phi_r (MN)}{M^r N^r} = \sum \frac{\delta^r{}_{12}}{\delta_1{}^r \delta_2{}^r} \mu(\delta_1) \mu(\delta_2)$$

(2) $$\sum \phi_r \left(\frac{M}{\delta_1}\right) \phi_r \left(\frac{N}{\delta_2}\right) \tau (\delta_{12}) = \left(\frac{MN}{g}\right)^r \cdot \sigma_r (g).$$

Here the summations extend over all divisors δ_1 of M, and δ_2 of N; and δ_{12} is the $g.\ c.\ d.$ of δ_1, δ_2, and g is the $g.\ c.\ d.$ of M, N.

1540. (S. SIVASANKARANARAYANA PILLAI):—Let (a, b) denote the greatest common divisor of a and b. Then (i) the necessary and sufficient condition that

$$a^{\phi(k)+1} \equiv a \pmod{k}$$

is that
$$\left(d, \frac{k}{d}\right) = 1, \text{ where } d = (a, k).$$

(ii) If $(a, k) = d$ and $\left(d, \frac{k}{d}\right) = 1$ and l is the least positive solution of

$$\left.\begin{array}{l} x \equiv 1 \left(\bmod \frac{k}{d}\right) \\ x \equiv 0 \pmod{d} \end{array}\right\}$$

then
$$a^{\phi(k)} \equiv l \pmod{k}.$$

1541. (S. SIVASANKARANARAYANA PILLAI):—If $d'(n)$ denotes the number of divisors of n, of the form $ar + b$ $(r = 0, 1, 2 \ldots)$, then

$$N(x) = \sum_{n=1}^{x} d'(n) = \frac{x \log x}{b} + \alpha x + O(\sqrt{x})$$

where α is a constant.

UNSOLVED QUESTIONS FOR WHICH SOLUTIONS ARE INVITED.

197. (K. J. SANJANA, M.A.):—If $N^{\frac{1}{2}}$ when converted into an infinite continued fraction has a recurring period of $2k$ partial quotients and if the krth convergent is denoted by C_r, prove that

$$C_{2r} = \tfrac{1}{2}\left\{\frac{N}{C_r} + C_r\right\}$$

and
$$C_{m+n} = \left\{\frac{N}{C_m} + C_n\right\} \div \left\{\frac{C_n}{C_m} + 1\right\}$$

where r, m and n are any integers.

210. (R. SRINIVASAN, M.A.):—A curve is drawn on a generated surface cutting every generating line at an angle θ and at a distance γ from the edge of regression measured along that generator. If the radius of curvature of the edge of regression at the point where it is touched by the generator varies as $\gamma^n \cos^2\theta$, find γ in terms of θ.

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1 American Journal of Mathematica, **51**, 1 (2 copies).

2 American Mathematical Monthly, **35**, 10 (3 copies). **36**, 1 (3 copies).

3 Annales de la'Ecole Normale Superieure, **63**, 8, 9, 10, 11, 12.

4 Astrophysical Journal, **68**, 5.

5 Bulletin des Science Mathematiques, **52**, Dec. 1928.

6 Japanese Journal of Mathematics, **5**, 3.

7 Journal fur die reine und angewandte Mathematik (Crelle), **160**, 1.

8 Mathematical Gazette, **14**, 198 (3 copies).

9 Mathematische Annalen, **100**, 4, 5.

10 Monthly Notices of the Royal Astronomical Society, **89**, 1, 2, 3.

11 Philosophical Magazine. **7**, 42, 43.

12 Philosophical Transactions of the Royal Society, **228**, 63 to 114, 115 to 150, 151 to 196.

13 Popular Astronomy, **37**, 1, 2.

14 Proceedings of the London Mathematical Society, **29**, 1, 2, 3, Second Series.

15 Proceedings of the Physico-mathematical Society of Japan, **10**, 9, 10, 11.

16 Proceedings of the Royal Society of London, **122**, 789, (2 copies) 790 (2 copies); **123**, 791.

17 Revue Semestrielle des Publications Mathematiques, **33**. 2nd Part.

18 Rendiconti del Seminario Mathematico e Fiseco, **2** (1928-VI).

Pamphlets.

1 Estudio de las Ciencias (Universidad Nacional de la Plata), No. 86.

2 A Review of Elementary Mathematics and Science, Edinburgh Mathematical Society, Mathematical Notes, No. 24.

3 Certain Configurations on Cubics, by Frank Chappell Ogg.

Numbers in black type refer to the volumes, and those in ordinary type to the numbers of the issues.

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